

A new perspective on renormalization: invariant actions, a dynamical DNA

Michael E. Glinsky*

CSIRO, 26 Dick Perry Ave, Kensington WA 6151, Australia and

School of Physics, University of Western Australia,

35 Stirling Hwy, Crawley WA 6009, Australia

(Dated: June 23, 2011)

Abstract

This is a note to a quantum field theorist, interpreting the “Mallat” transformation (i.e., group invariant scattering) [1]. First the characteristics of the transformation are examined, and related to some well known issues with renormalization. The Lagrangian density and action for the texture ID problem that is the focus of Mallat is derived. This is the key translation in order to make the correspondence to the mathematics of quantum field theory and path integrals. Next Mallat’s transformation is developed from the perspective of a “renormalization” of the actions. That is, a change of coordinates to “smoothed” paths. The effective action is then calculated utilising a stationary phase approximation to the usual Gaussian integrals that are encountered. It should be noted that this evaluation is only the leading term in an asymptotic series. The fundamental excitations of the system are now well identified through their relationships to the generating function and the corresponding currents. The resulting Feynman diagrams are presented. The difference between two states of the system are then quantified by excitations needed to scatter one state into the other. The renormalization is based on an iterative wavelet transformation where the wavelet family respects the group symmetries of both the base manifold and the field. Finally, a connection to the analysis of complex nonlinear systems will be made where the difference in these states of complex systems will form a natural metric.

PACS numbers: 05.10.Cc, 11.10.Gh

*Electronic address: michael.glinsky@csiro.au

Background

Lets start with Mallat's observations about the shortcomings of Fourier and wavelet transforms [1]. While Fourier is time invariant, it is not Lifshitz continuous to small changes in the signal at small scales. This is just the well known ultraviolet divergence at small scales in renormalization. Wilson style renormalization [2] is based on band limited Fourier transforms. Although wavelet transforms are Lifshitz continuous, they are not time invariant. Mallat's solution is to iteratively take wavelet transforms until the result is time invariant. There is a strange, but necessary, *ansatz* that he makes of taking the modulus between each wavelet transform. He orders the transformation in terms of how many wavelets are interacting (order of the interaction) and the scale of the wavelets. There is a second hierarchy in terms of the wavelet family.

As a quantum field theory physicist, one has certain bells going off. Words come to mind like Wick ordering, invariant actions, stationary phase, and field versus space quantization.

Since it is well known that many problems like data assimilation, finance, biology, statistical mechanics and quantum field theory can be put in a path integral formulation, we set out to cast Mallat's problem [3] and method in this language.

Translation of Mallat's problem to quantum field theory

Mallat considers an image $f(\vec{x})$ on \mathbb{R}^2 deformed by $\vec{\tau}(\vec{x})$ such that $f(\vec{x}) \rightarrow f(\vec{x} - \vec{\tau}(\vec{x}))$. Look at this as a two dimensional field $\vec{x}(t)$ subjected to dynamical flow ϕ_t defined by the vector field given by $\vec{\tau}(\vec{x}_o)$ and $\overleftarrow{\nabla} \vec{\tau}(\vec{x})$ such that

$$\vec{\tau}(\vec{x}) = [\overleftarrow{\nabla} \vec{\tau}(\vec{x}) - [\overleftarrow{\nabla} \vec{\tau}(\vec{x}_o) + \overleftarrow{I}]] \cdot \vec{\tau}(\vec{x}_o) \quad (1)$$

and t_o is given by $\phi_{t_o}(\vec{x}_o) = \vec{\tau}(\vec{x}_o)$ and $\phi_o(\vec{x}_o) = \vec{x}_o$. One can define an action [4] [5] as

$$S[\vec{x}(t)] = \int_0^{t_o} L(\vec{x}(t), \dot{\vec{x}}(t)) dt \quad (2)$$

where

$$L(\vec{x}(t), \dot{\vec{x}}(t)) = [\dot{\vec{x}} - \vec{\tau}(\vec{x})] \cdot \vec{g}(\vec{x}) \quad (3)$$

and $\vec{g}(\vec{x})$ is an arbitrary gauge. In fact the flow is Hamiltonian with $H(\vec{x}, \vec{p}) = \vec{\tau}(\vec{x}) \cdot \vec{p}$, giving Hamilton's equations

$$\dot{\vec{x}} = \vec{\tau}(\vec{x}) \quad (4)$$

$$\dot{\vec{p}} = -\vec{p} \cdot \overleftarrow{\nabla} \vec{\tau}(\vec{x}). \quad (5)$$

The image is just the wave function so that

$$f(\vec{x}, t) = \int e^{iS[\vec{x}(t)]/\hbar} \mathcal{D}[\vec{x}(t)] f(\vec{x}, 0) \quad (6)$$

which reduces to

$$f(\vec{x}, t) = f(\vec{x} - \phi_t(\vec{x})) = f(\vec{x} - \vec{\tau}(\vec{x})) \quad (7)$$

in the classical limit. Having reduced Mallat's problem to a path integral with a 2D field on a 1D "space" coordinate, t , or base manifold; we can now move to understanding this transformation.

Mallat's invariant scattering as an iterative renormalization

For simplicity, we first consider a 1D time dependent Lagrangian $L(x, \dot{x}, t)$, with a 1D field, $x(t)$, on a 1D coordinate, t (that is, 1D base manifold, \mathbb{R}). We will generalize at the end to an N dimensional field on a M dimensional base manifold. This will highlight why there is a second hierarchy associated with the wavelet family. The first hierarchy is with respect to the wavelet scale. To keep things tractable we will also consider the discrete time sliced version. The action is

$$S_o[\{x_j\}] = \sum_j L(x_j, \dot{x}_j, t_j) \Delta t \quad (8)$$

We now start an iterative "renormalization" by a change of coordinate to the wavelet basis (this will then be iterated). The "renormalized" smoothed coordinates are

$$\bar{x}_{ik} = \sum_j x_j W_{ik}(t_j) \quad (9)$$

Where W_{ik} is the mother wavelet with scale, i and time, k . This is a complex orthogonal transformation such that

$$\delta_{ii'}\delta_{kk'} = \sum_j W_{ik}(t_j)W_{i'k'}(t_j) \quad (10)$$

$$\left| \frac{\partial\{x_j\}}{\partial\{\bar{x}_{ik}\}} \right| = 1 \quad (11)$$

$$x_j = \sum_{ik} \bar{x}_{ik} W_{ik}(t_j) \quad (12)$$

$$\dot{x}_j = \sum_{ik} \bar{x}_{ik} W'_{ik}(t_j) \quad (13)$$

In general the wavelet family should respect the group symmetries of both the base manifold and the field, that is symmetries of the action functional.

The question that we now address is “what is the effective renormalized action?” We first recognize that due to the orthonormality of wavelet transforms that there is a simple mean field expression for

$$\frac{\partial S_o[\{\bar{x}_{ik}\}]}{\partial \bar{x}_{ik}} = \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} + W'_{ik}(t_j) \frac{\partial L}{\partial v} \quad (14)$$

$$= \left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} + \left\langle \frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) \right\rangle_{ik} \quad (15)$$

$$= \langle \nabla L \rangle_{ik} + \langle \dot{p} \rangle_{ik} \quad (16)$$

where we have used integration by parts to move the time derivative from W'_{ik} to $\partial L/\partial v$, have used the definitions

$$\left\langle \frac{\partial L}{\partial x} \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j) \frac{\partial L}{\partial x} \quad (17)$$

$$\left\langle \frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) \right\rangle_{ik} \equiv \sum_j W_{ik}(t_j) \frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) \quad (18)$$

to expand $\partial L/\partial x$ and $(d/dt)(\partial L/\partial v)$ in the wavelet basis, and have set $\Delta t = \hbar = i = 1$. We get the expression for the effective action [6] by considering the generating function

$$C[\{J_{ik}\}] = \ln \left[\int \exp \left(-S_o[\{\bar{x}_{ik}\}] + \sum_{ik} J_{ik} \bar{x}_{ik} \right) \prod_{ik} d\bar{x}_{ik} \right] \quad (19)$$

Here J_{ik} are the currents which generate the fundamental excitations of the system. The effective action is just the Legendre transform of the generating function C

$$S[\{\varphi_{ik}\}] = -C[\{J_{ik}\}] + \sum_{ik} J_{ik} \varphi_{ik} \quad (20)$$

We separate the action into two parts $S = S_o + S_1$ and expand the integral in terms of

$$\psi_{ik} \equiv \bar{x}_{ik} - \varphi_{ik} \ll \varphi_{ik} \quad (21)$$

to second order in ψ

$$\exp[S_1[\{\varphi\}]] = \int \exp \left[\sum_{ik} \frac{\partial S_1}{\partial \varphi_{ik}} \psi_{ik} - \frac{1}{2} \sum_{iki'k'} \psi_{ik} \frac{\partial^2 S_o}{\partial \varphi_{ik} \partial \varphi_{i'k'}} \psi_{i'k'} \right] \prod_{ik} d\psi_{ik} \quad (22)$$

This is a standard Gaussian integral that can be done but the contour must be deformed so that the phase of the integrand is constant. This is equivalent to taking the modulus of \bar{x}_{ik} , giving a natural justification to the *ansatz* of taking the modulus after each wavelet transform in Mallat's method. One gets the equation for S_1 in terms of

$$\overleftarrow{\gamma} = \frac{\partial^2 S_o}{\partial \varphi_{ik} \partial \varphi_{i'k'}} = \frac{\partial \langle \nabla L \rangle_{ik}}{\partial \psi_{i'k'}} + \frac{\partial \langle \dot{p} \rangle_{ik}}{\partial \psi_{i'k'}} \quad (23)$$

$$2S_1 = \Delta - \sum_{iki'k'} \frac{\partial S_1}{\partial \varphi_{ik}} \overleftarrow{\gamma} \frac{\partial S_1}{\partial \varphi_{i'k'}} \quad (24)$$

where

$$\Delta \equiv \ln \left| \frac{\gamma}{2\pi} \right|. \quad (25)$$

This can be solved giving $S[\{\varphi_{ik}\}]$. The mean path $\langle \varphi_{ik} \rangle$ is given by

$$\left. \frac{\partial S}{\partial \varphi_{ik}} \right|_{\langle \varphi_{ik} \rangle} = 0, \quad (26)$$

giving the expansion about $\langle \varphi_{ik} \rangle$

$$S = S_o[\{\langle \varphi_{ik} \rangle\}] + \frac{1}{2} \sum_{iki'k'} (\varphi_{ik} - \langle \varphi_{ik} \rangle) \frac{\partial^2 S_o}{\partial \varphi_{ik} \partial \varphi_{i'k'}} (\varphi_{i'k'} - \langle \varphi_{i'k'} \rangle) \quad (27)$$

$$+ \mathcal{O} [(\varphi_{ik} - \langle \varphi_{ik} \rangle)^3]. \quad (28)$$

Now as the wavelet transformation is repeated it is a conjecture that the second order term diagonalizes and the 3rd and higher order terms goes to zero. A way to see this is to assume one of the off diagonal terms is not zero. This is a statement that there is an interaction between two wavelets of different scales. But, this excitation is already on the diagonal. The same is true of the 3rd order terms. Therefore the action takes the simple form

$$S[\{\varphi_p\}] = S_o[\{\langle \varphi_p \rangle\}] + \frac{1}{2} \sum_p \frac{\partial^2 S_o[\{\langle \varphi_p \rangle\}]}{\partial \varphi_p^2} (\varphi_p - \langle \varphi_p \rangle)^2 \quad (29)$$

Here the path, p , is specified by the order of the interaction, N_s , and a set of scales $\{s_n\}$. In other words

$$p = (N_s; s_1, s_2, \dots, s_{N_s}). \quad (30)$$

In the $|\varphi_p\rangle$ coordinate system, this leads to the simple form of the measure which separates

$$d\mu = \frac{1}{Z} \prod_p \exp [S_p (\varphi_p - \langle \varphi_p \rangle)^2] d\varphi_p \quad (31)$$

where

$$S_p \equiv \frac{1}{2} \left. \frac{\partial^2 S_o}{\partial \varphi_p^2} \right|_{\langle \varphi_p \rangle} \quad (32)$$

and

$$\left. \frac{\partial S_o}{\partial \varphi_p} \right|_{\langle \varphi_p \rangle} = 0. \quad (33)$$

The dispersion relation is S_p . It also gives the energy of the fundamental excitations about the mean states $\langle \varphi_p \rangle$. These invariant actions, S_p , characterize the dynamics specified by S_o and $|\varphi_p\rangle$ is the corresponding basis. This is why Mallat's method is working so well. He is projecting the image onto a basis where the dynamics is diagonal and specified only by S_p .

S-matrix, stationarity, restricted ergodicity, and the probability density function

There is an important connection that we must make to the general scattering problem and the probability density function. We start by writing the familiar form [5] for the path integral propagator, U ,

$$f(x_f, t_f) = \int e^{S_o[\{x_j\}]} \mathcal{D}[\{x_j\}] f(x_o, t_o) \quad (34)$$

$$= U(x_f, t_f; x_o, t_o) f(x_o, t_o) \quad (35)$$

so that

$$U(x_f, t_f; x_o, t_o) \equiv \int e^{S_o[\{x_j\}]} \mathcal{D}[\{x_j\}] \quad (36)$$

The scattering operator, \hat{S} , or S-matrix [6] [4] is defined as

$$\hat{S} \equiv \lim_{\substack{t_o \rightarrow -\infty \\ t_f \rightarrow \infty}} U(x_f, t_f; x_o, t_o) = \lim_{\substack{t_o \rightarrow -\infty \\ t_f \rightarrow \infty}} \int e^{S_o[\{x_j\}]} \mathcal{D}[\{x_j\}]. \quad (37)$$

This is a unitary operator that contains all the dynamical information.

Now if we construct the renormalization of the path $\{x_j\}$ such that the wavelets have the same group symmetries of the base manifold and of the field (remember that an important part of constructing an action functional is ensuring that it has the expected symmetries of the field, and vice versa), it was shown in the previous section that,

$$\hat{S} = \lim \int e^{S_o[\{x_j\}]} \mathcal{D}[\{x_j\}] = \lim \int e^{S_{eff}[\{\varphi_p\}]} \mathcal{D}[\{\varphi_p\}] \quad (38)$$

$$\approx \lim \int e^{S_o[\{\varphi_p\}]} \prod_p \exp [S_p(\varphi_p - \langle \varphi_p \rangle)^2] d\varphi_p \quad (39)$$

to the leading order asymptotic term. This is to say that the S-matrix is diagonalized in the $|\varphi_p\rangle$ basis. The limit simply takes p to infinite scale or equivalently to the continuous limit of $\Delta t \rightarrow 0$. S_p can be identified as the invariant actions of the motion, that is the dispersion relation.

We now allow $f(x, t = 0)$ to evolve according to the dynamics for a long enough period of time so that the motion can ergodically visit all parts of the dynamical attractive surfaces. Then $f(x, t \rightarrow \infty)$ will have reached a stationary state $f_\infty(x)$. Even though the dynamics is not ergodic, it is ergodic in a restricted sense, that is within a $|\varphi_p\rangle$ mode the time average will equal the field average. To be more concrete, let $f(x_{kj}) = f(x_k(t_j)) = f(x, t)$. Recognize that $\langle f|\varphi_p\rangle \equiv f_{p_j}(k)$. By the restricted ergodicity $f_{p_j}(k) = f_{p_k}(j)$, and by time stationarity $f_{p_k}(j) = f_p$ is a constant. Together these imply stationarity of f with respect to the field so that $\langle f|\varphi_p\rangle = f_{p_k}(j) = f_p$. Eventhough the transform is with respect to the time evolution, one can calculate the projection of a stationary f onto $|\varphi_p\rangle$ by taking the transform of $f_\infty(x)$ with respect to the field, x . The stationary wave function can be directly related to the probability density function, $\rho(x) = f_\infty(x)f_\infty^*(x)$. Expectation values can be written as

$$\langle A(x) \rangle = \int A(x)\rho(x)dx \quad (40)$$

In the classical limit

$$\rho(x) = \sum_{jk} f(x_k, t = 0)f^*(x_k, t = 0)\delta(x - x_{cl}(x_k, t_j)). \quad (41)$$

Second quantization of the field, scattering and Feynman diagrams

We now use this $|\varphi_p\rangle$ basis to second quantize the field [4] [7]. This is a change of basis to the ‘‘action’’ basis, sometimes called the energy basis, where the destruction operators,

a_p , and creation operators, a_p^\dagger , are defined such that

$$[a_p, a_{p'}^\dagger] = \delta_{pp'}, \quad (42)$$

$$[a_p^\dagger, a_{p'}^\dagger] = 0, \quad (43)$$

$$[a_p, a_{p'}] = 0, \quad (44)$$

$$S = S_o + \sum_p S_p a_p^\dagger a_p \quad (45)$$

and an arbitrary state, $|f\rangle$, can be expanded in the basis such that

$$|f\rangle = \prod_p \left(\frac{(a_p^\dagger)^{N_p}}{(N_p!)^{1/2}} \right) |\varphi_o\rangle, \quad (46)$$

$$|\varphi_p\rangle = \frac{(a_p^\dagger)^{N_p}}{(N_p!)^{1/2}} |\varphi_o\rangle, \quad (47)$$

$$|\langle \varphi_p | f \rangle|^2 = \frac{N_p}{\sum_p N_p}, \quad (48)$$

$$\langle \varphi_p | S | \varphi_p \rangle = S_o + N_p S_p, \quad (49)$$

$$\langle f | S | f \rangle = S_o + \sum_p N_p S_p. \quad (50)$$

The projection of $|f\rangle$ onto the $|\varphi_p\rangle$ basis is

$$\sum_p |\varphi_p\rangle \langle \varphi_p | f \rangle = MW_t \dots MW_t f(x(t), t) \quad (51)$$

$$= MW_x \dots MW_x f_\infty(x) \quad (52)$$

$$= S f_\infty = \sum_p |\varphi_p\rangle S(p) f_\infty(x), \quad (53)$$

where M is the modulus operator, W_x is the wavelet transform with respect to x , W_t is the wavelet transform with respect to t , and $S(p)$ is Mallat's scattering operator. Note that we have used the results of the previous section. We can now identify $\langle \varphi_p | f \rangle = S(p) f_\infty(x)$ for the case of stationary f . Since $|\varphi_p\rangle$ are the modes associated with the fundamental excitations of the system, a change in the projection of the wave function onto this basis can only be changed by an interaction with an external field. The natural metric of this

external interaction, scattering a state $|f\rangle$ into a state $|g\rangle$, is given by

$$d^2(f, g) = \sum_p |\langle \varphi_p | f \rangle - \langle \varphi_p | g \rangle|^2 \quad (54)$$

$$= \sum_p |S(p)f - S(p)g|^2 \quad (55)$$

$$\propto \sum_p \left(\Delta \sqrt{N_p} \right)^2, \quad (56)$$

where the proportionality is only valid if the total number, $\sum_p N_p$, is conserved.

We now move on to the generalisation of this method to a N dimensional field on a M dimensional space. We pay particular attention to the structure of the path, p . In terms of the space coordinate like t , the path is related to the scales of the group parameter, t . In this case it is an affine group, or Lie group, associated with the flow (given by the vector field) on phase space. It is divided into order according to how many scales have interacted, or in Mallat's language, how many scale scatterings have taken place. One needs to know the order of the path or state, N_s (that is, how many scales of which it is composed), and what the scales are, $\{s_n\}$.

In addition, for fields of greater than one dimension there is a second heirachy associated with the group of a transformation of the field. The subpath is related to the scales, λ , of the group parameter of the field transformation. One needs to know the order of this substate, N_{λ_n} (that is, how many scales of which it is composed), and the scales, $\{\lambda_{mn}\}$.

To be more graphic, we call an elementary excitation of this system a *genton*. Its corresponding creation operator is a_p^\dagger and it is indexed by p . This creation operator can be expanded as

$$a_p^\dagger = a_{p_o}^\dagger \prod_{n=1}^{N_s} a_{s_n}^\dagger, \quad (57)$$

where N_s is the order of the state and $s_n > \dots > s_1$. The final creation operator, $a_{p_o}^\dagger$ is for an infinite scale or, in other words, for a state which has no scale interactions (that is, free). The excitation associated with $a_{s_n}^\dagger$ we call an *indyon*. We call it this because it is associated with the scale of the Lie group parameter, which are independent coordinates of which the field is a function. We further expand the creation operator as

$$a_p^\dagger = a_{p_o}^\dagger \prod_{n=1}^{N_s} a_{s_n, o}^\dagger \prod_{m=1}^{N_{\lambda_n}} a_{s_n, \lambda_{mn}}^\dagger, \quad (58)$$

where N_{λ_n} is the order of the subpath and $\lambda_{N_{\lambda_n}n} > \dots > \lambda_{1n}$. The excitation associated with the $a_{s_n, \lambda_{mn}}^\dagger$ operator we call a *depton*. We call it this because it is associated with the scale of the group transformation of the field – the dependant field parameters.

We now construct the Feynman diagram for the interaction of the two systems with states $|f_1\rangle$ and $|f_2\rangle$. They evolve and interact according to the action

$$S = S_{o1} + \sum_{p_1} S_{p_1} a_{p_1}^\dagger a_{p_1} + S_{o2} + \sum_{p_2} S_{p_2} a_{p_2}^\dagger a_{p_2} + \sum_p C_p a_{p_2=p}^\dagger a_{p_1=p} \quad (59)$$

A simple view of the interaction is shown by the Feynman diagram in Fig. 1. It is a scattering

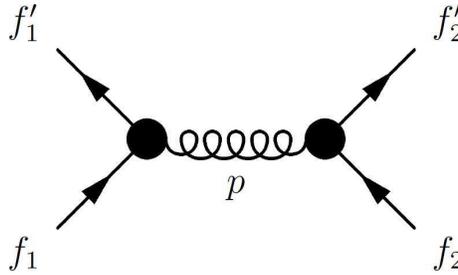


FIG. 1: Feynman diagram of scattering of wavefunction, f_1 , by another wavefunction, f_2 , mediated by a fundamental excitation, *genton*, of the system, or in other words a quasi-particle.

of the state f_1 into f'_1 by the emission of a *genton* of state p . There is a corresponding scattering of the state f_2 into f'_2 by the absorption of the same *genton*.

This interaction, or scattering, can be further expanded by including the *indiyons*, as shown in Fig.2. Note that for this example, it is a third order scattering of this *genton*, evidenced by the emission of three *indiyons* at once from the vertex where the *genton* is scattered from a state p to p_o .

A final expansion of this picture is yielded by including the scattering of the *indiyons* by the *deptons*, as shown in Fig. 3. Again note the order of the *indiyon* scattering (3, 1 and 2 respectively) evidenced by the emission of *deptons* from the vertices where the *indiyons* are scattered from a state s_n into $s_{n,o}$. The legend for these Feynman diagrams is shown in Fig. 4.

These Feynman diagrams complete the connection to the language that Mallat uses. The elementary excitation can be viewed as scatterings. They are generated by the group symmetries of both the independent coordinates and the dependant fields – they are group

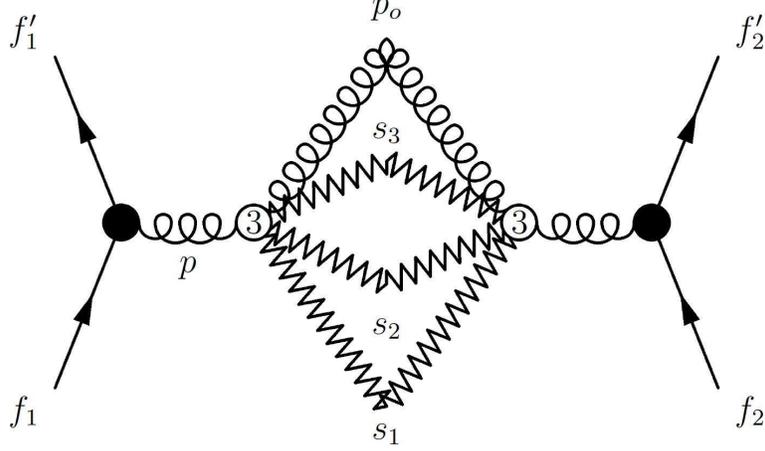


FIG. 2: Feynman diagram of the scattering of Fig. 1 expanded into the component *indyons* associated with the independent parameters of which the fields are functions. The *genton* is of order 3.

invariant scatterings. The scattering of a wave function $|f\rangle$ into a wave function $|g\rangle$ by the absorption of a *genton* of path p is given by

$$|g\rangle = a_p^\dagger |f\rangle \quad (60)$$

The operator a_p^\dagger is simply the scattering operator of Mallat with the simple picture given by Fig. 5.

For the specific case of phonons, the *indyons* are related to quantization of the wave vector, \vec{k} , and the *depton* is related to the quantization of the polarization (orientation of the field) The generic excitation, *genton*, is the the phonon.

A more compact graphic representation of the generic excitation is shown in Fig. 6. It shows a *genton* with the same structure as shown in Fig. 3. Instead of each component excitation being shown as a line, it is reduced to a point particle. This allows the structure of the *genton* to be more easily seen.

Conclusions

In summary, we fell that Mallat's method can be looked upon as an iterative, wavelet based, renormalization. We know that this is different from what one normally thinks about in renormalization, but goes back to the fundamental calculation of Wilson. Wilson made a

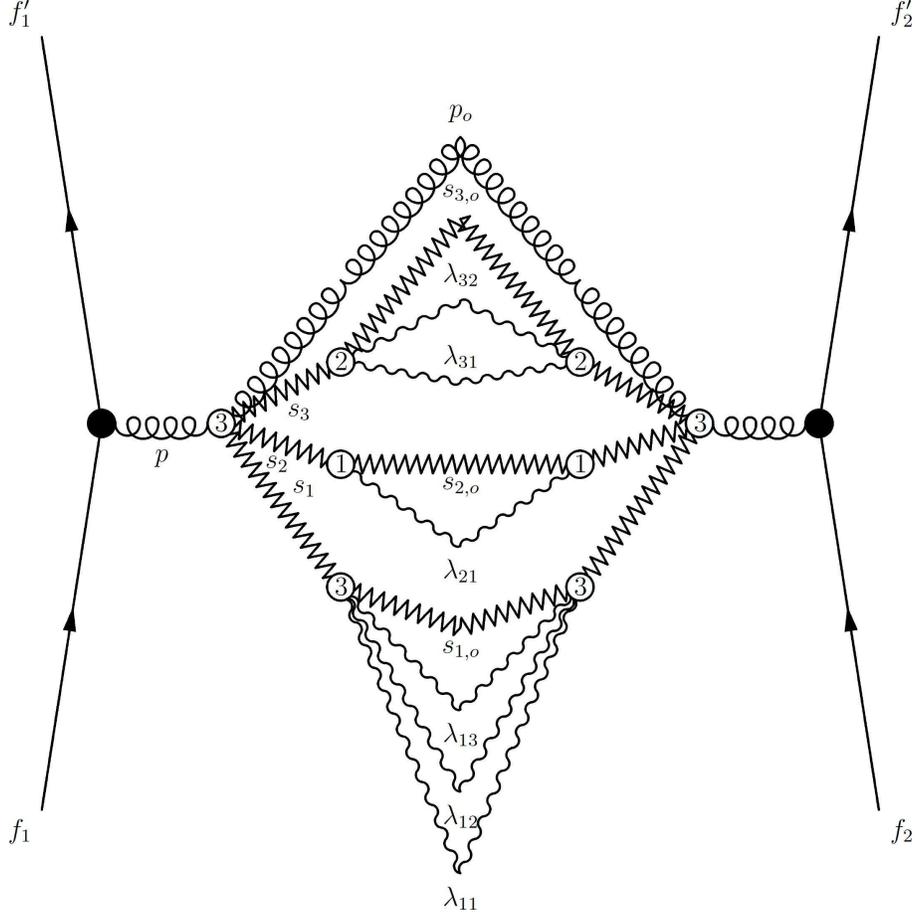


FIG. 3: Feynman diagram of the scattering of Fig. 1 expanded into the component *depton*s associated with the dependant field parameters. This *genton* is scattered by three *indyons* of order 3, 1, and 2 respectively.

change of variable to a “smoothed” coordinate using a band limited Fourier transform, and then calculated the “smoothed” action. He gave up on this direct approach because he did not have the wavelet transform (especially the iteration) in his mathematical toolbox. The following quote from his seminal 1971 Physics Review article [2] is quite illuminating in that he stated his desire for such a set of wave packets: *“This is a quantitative characterization of a complete orthonormal set of minimal wave packets. For quantitative purposes one would have to take into account tails of the wave packets which extend outside their assigned cells. It will be assumed here that one can divide phase space into cells of unit volume in any way one pleases and still be able to construct a corresponding set of minimal wave packets. There is no guarantee that this is actually possible, and no examples of such a set of wave packets*

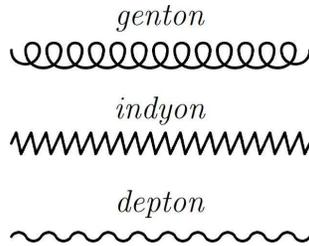


FIG. 4: Legend for feynman diagrams.

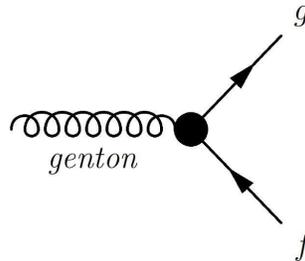


FIG. 5: Simple single scattering of a wavefunction f into a wavefunction g by absorption of a *genton*.

will be given here.”

The form of the effective action is extremely simple in this basis, where only simple excitations need to be taken into account, no coupling. The dynamics is then specified by these excitation energies, or actions, S_p . This is to say, the dynamics is fully decoupled. These actions, S_p , are the unique finger print of the dynamics much like DNA is for a biological organism – it is *dynamical DNA*.

Some comments on the direct relationship of this method to the language of Mallat. His seminal paper is entitled “Group Invariant Scattering”. The basis that his transformation projects a probability density function onto is a basis of states that diagonalizes the scattering operator – the S-matrix. This is a unitary operator that contains all the dynamical information about the states. The transformation is built to respect the group symmetries of both the base manifold and the field. Therefore the projection of the probability density function onto this basis gives the occupation numbers in terms of these group symmetries. Two different states will therefore have a natural metric with respect to this group invariant scattering in terms of the number and types of fundamental excitations required to scatter one state into the other. This does assume that both states are governed by dynamics with

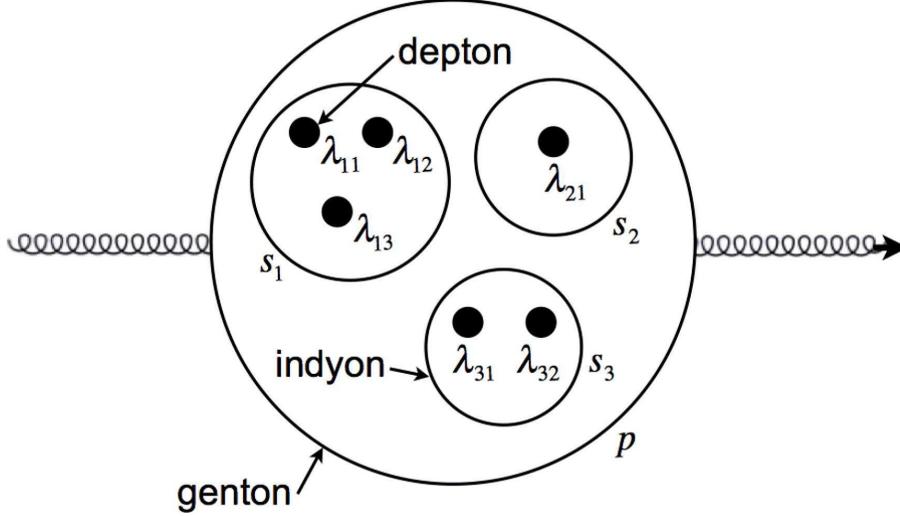


FIG. 6: A representation of the generic excitation, *genton*, made up of *indyons* and *deptons*.

the same group symmetries.

One may well be astonished that a general basis can diagonalize the S-matrix of any system. After all, the eigen basis is the solution to the dynamics. How can every problem have the same solution? The answer is that it does not. Remember that the wavelet family must respect the group symmetries of the dynamics. A very important part of constructing the action functional, which determines the dynamics of the system, is making sure that it has all the appropriate symmetries. This basis is therefore not so general, it must match the symmetries of your system.

Finally, we would like to point out a major advantage of this renormalized coordinate system. It provides a very compact coordinate system for analyzing the dynamics of complex nonlinear systems. The number of important coordinates will be directly related to the finite dimension of the dynamical attractors, a countable finite number. The important coordinates will be the ones of lowest order of interaction in the hierarchy, so they will be easy to find. The metric that these coordinates give allows the similarity or difference of states to be quantified.

Acknowledgements

The author would like to thank Stephane Mallat for many useful discussions about his transformation, CSIRO Office of the Chief Exectutive for funding this research, and The

Santa Fe Institute and The University of Western Australia for their hospitality.

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