① MHD Eq approx
② Helicity invariant ← $K(\alpha,\beta)$ $\nabla \times B = \lambda(\alpha,\beta)\vec{B}$
K $\nabla \times B = \lambda \vec{B}$
③ what is helicity
④ Real Decay $\dot{W}$ vs $\dot{K}$ Relaxed states
Pinch
⑤ Taylor Equilibrium BFM
→ relaxed B_φ, B_θ
→ current limitation < two modes
⑥ Multi-pinch < why see limitation better
⑦ Non-linear stability

use multi modes

Theory

① MHD Eq + approx • $\vec{B}$ moves with fluid
• eq $J \times B = \nabla P$ $B \cdot \nabla P = 0$
② Helicity • define $J \cdot \nabla P = 0$
• $dH/dt = 0$
• Describe
③ Relaxed states
• Decay $\dot{W}$ vs $\dot{K}$ $B \cdot \nabla \lambda = 0$
• ① $\nabla \times B = \lambda(\alpha,\beta)\vec{B}$ $\frac{d\tilde{W}}{d\lambda} = 0$ or $\delta \int B^2 = 0$ $K(\alpha,\beta) = $ const $\forall \alpha,\beta$
② $\nabla \times B = \mu B$ $\delta \int B^2 = 0$ K const
• non-linear stability

Pinch
① Describe Geometry define Θ
② Relaxed state
• B_φ, $B_\theta(r,\Theta)$
• current limitation
③ Multi pinch • two modes
• limitation

Helicity Conservation and RFP

Theory

- MHD
- Helicity
- Taylor Relaxation

RFP

- Relaxed State
  (circular)
  → Current limitation
  → (F, θ) diagram
  $B_\theta(r)$ $B_\varphi(r)$ profiles
- Multipinch

$\boxed{\text{Theory}}$

assume <u>Ideal</u> <u>MHD</u>

$$\left(\begin{array}{l} L \to \infty \\ \omega \to 0 \\ n_e \approx n_i \end{array} \right)$$

variables : $\rho_m, \vec{v}, \vec{J}, \vec{E}, \vec{B}, p$ (14)

equations :

(1) $\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}) = 0$ Mass Continuity

(3) $\rho_m \frac{\partial \vec{v}}{\partial t} = \frac{\vec{J} \times \vec{B}}{c} - \nabla p$ Momentum Eq

(3) $\vec{E} + \frac{\vec{v} \times \vec{B}}{c} = \frac{\vec{J}}{\sigma}$ Ohm's Law

$$\left. \begin{array}{ll} (3) & \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ (3) & \nabla \times \vec{B} = \frac{4\pi \vec{J}}{c} \end{array} \right\} \text{Maxwell's Equations}$$

(1) Equation of State

three possibilies $\left\{ \begin{array}{l} \nabla \cdot v = 0 \\ \frac{d}{dt}(p \rho_m^{-\gamma}) = 0 \\ \frac{d}{dt}\left(\frac{p}{\rho_m}\right) = 0 \end{array} \right.$

$\nabla \cdot B = 0$ (initial condition)

$\nabla \cdot E = 0$ (not used because $n_e \approx n_i$)

Helicity

$$\text{helicity} \equiv K \equiv \int_{V} \vec{A}\cdot\vec{B}\, d^3V$$

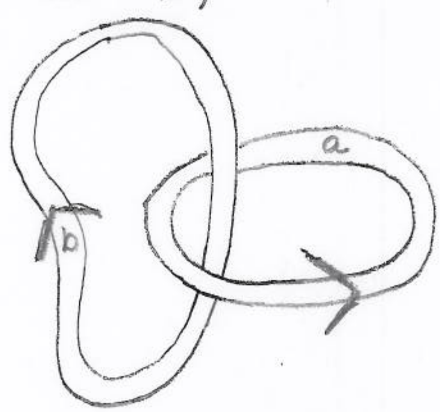
V = flux tube

($i.e.$, $\vec{B}\cdot\hat{n}=0$ on ∂V)

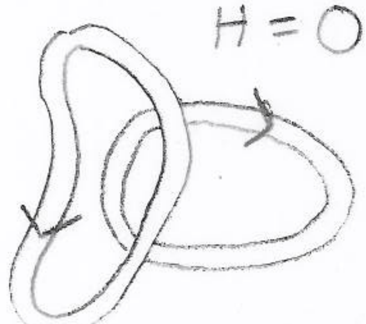
$$= \lim_{\substack{N\to\infty \\ \delta\varphi\to 0}} \sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}(\delta\Phi)^2$$

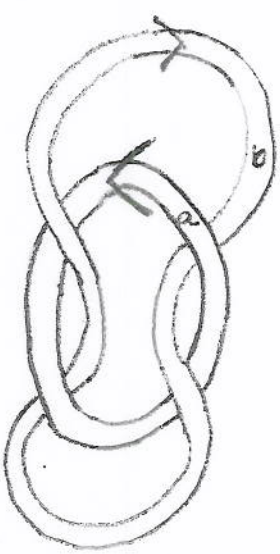
L_{12} = # of linkage of flux tubes

$H = -(2\,\Phi_a\,\Phi_b)$



$H = 0$





$H = +2(2\,\Phi_a\,\Phi_b)$

- If $\vec{B}\cdot\hat{n}=0$ on ∂V

  $\Delta K=0$ under gauge tranform $\vec{A}\to\vec{A}+\nabla\chi$

- If $\vec{B}\cdot\hat{n}=0$ on ∂V

  and $\vec{E}+\frac{\vec{v}\times\vec{B}}{c}=0$ (i.e., $\sigma=\infty$)

$$\frac{d}{dt}\left[\int\vec{A}\cdot\vec{B}\right]=0=\frac{dK}{dt}$$

$$\boxed{\text{Helicity contained in a flux surface is a constant of the motion}}$$

in other words, the linkage of the lines of $\vec{B}$ is a topological invariant of the motion.

Relaxation

- Decay of energy vs. helicity

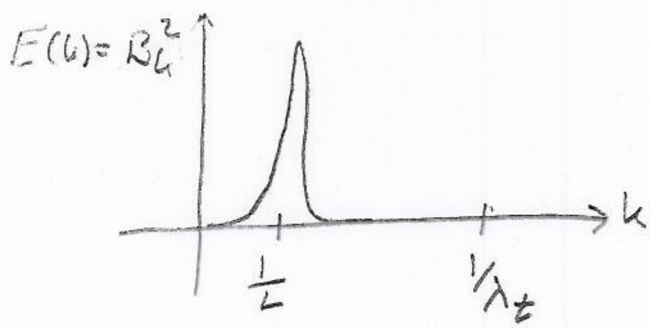
$$\dot{K} \sim -2\eta \int \vec{J} \cdot \vec{B} \lesssim -2\eta \sum_k k B_k^2$$

Cauchy-Schwarz
fourier decomposition (spacial)

$$\dot{W} \sim -\eta \int J^2 \sim -\eta \sum_k k^2 B_k^2$$

fourier decomposition

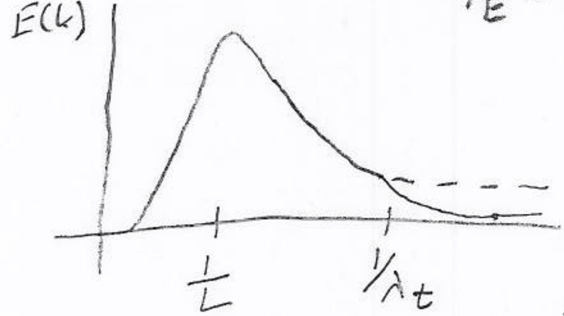
→ no turbulence



resistive time scale
$\tau_E \sim 10$ msec

→ turbulence (assume inertial or steady state)

$\tau_E \sim 10 \mu$sec



$$E(k) = K_0 e^{-(k\lambda_t)^{4/3}} \varepsilon^{2/3} k^{-5/3}$$

ε = pump rate

$$\frac{\tau_W}{\tau_K} \lesssim \left(\frac{\lambda_t}{L}\right)\left(\frac{L\omega_0}{K_0}\right) << 1$$

Taylor's Conjecture

$\frac{\tau_w}{\tau_K} << 1 \Rightarrow$ The system relaxes to a state of minimum energy subject to the constraint $K = \text{const}$ and $\vec{B}\cdot\hat{n} = 0$ on ∂V

$$W = \int (\nabla\times\vec{A})^2 \, d^3V \qquad K = \int \vec{A}\cdot(\nabla\times\vec{A}) \, d^3V$$

use lagrange multiplier and set

$$\delta \int_V (\nabla\times\vec{A})^2 - \mu \vec{A}\cdot(\nabla\times\vec{A}) \, d^3V = 0$$

$$\text{with } (\nabla\times\vec{A})\cdot\hat{n}\Big|_{\partial V} = 0$$

gives

$$\nabla\times\vec{B} = \mu\vec{B} \qquad \text{with} \begin{cases} B\cdot\hat{n}\big|_{\partial V} = 0 \\ K = \text{const} \\ \phi = \text{toroidal flux} = \text{const} \end{cases}$$

this is in contrast to $\sigma = \infty$ $\nabla p = 0$ MHD equilibrium

$$\nabla\times\vec{B} = \lambda(\vec{x})\vec{B} \quad \text{with } \vec{B}\cdot\nabla\lambda = 0$$

• Stability

$$\Delta W(\delta\vec{A}) = \delta^2 \int \left[(\nabla\times\vec{A})^2 - \mu \vec{A}\cdot(\nabla\times\vec{A}) \right] d^3V$$
$$= \int \left[(\nabla\times\delta\vec{A})^2 - \mu\, \delta\vec{A}\cdot(\nabla\times\delta\vec{A}) \right] d^3V$$

now minimize ΔW subject to the constraint $\delta\vec{A}\big|_{\partial V} = 0$ and $\int (\nabla\times\delta\vec{A})^2 = 1$

use method of lagrange multipliers

$$\delta \int \left[(\nabla\times\delta\vec{A})^2 - \frac{\mu}{1-g}\, \delta\vec{A}\cdot(\nabla\times\delta\vec{A}) \right] d^3V = 0$$
with $\delta\vec{A}\big|_{\partial V} = 0$

$$\nabla\times(\nabla\times\delta A) = \frac{\mu}{1-g}(\nabla\times\delta\vec{A})$$

gives

$$(\Delta W)_{min} = g \int (\nabla\times\delta\vec{A})^2 d^3V = g$$

if $\mu \leq \mu_{min}$ = minimum eigen value

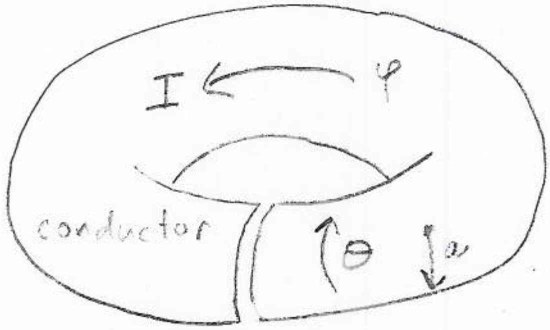
$\Rightarrow g \geq 0$

$\Rightarrow (\Delta W)_{min} \geq 0 \quad (\forall\, \delta\vec{A})$

$\Rightarrow$ $\boxed{\text{solution is true minimum not a min-max}}$

RFP

Geometry



K, ψ constant

$$K = -\psi \cdot \left(c \int V\, dt \right)$$

ψ = toroidal flux $= \pi a^2 B_0$

V = induced toroidal EMF

I = toroidal current

B_0 = initial average toroidal field

Define

$$\Theta \equiv \text{pinch ratio} \equiv \frac{2I}{ac B_0} = \frac{\mu a}{2} = \frac{B_\theta(\text{wall})}{B_0}$$

$$F \equiv \text{field reversal ratio} \equiv \frac{B_\varphi(\text{wall})}{B_0}$$

Relaxed State

Solution of $\nabla \times B = \mu \vec{B}$ for cylinder is

$$\vec{B} = \alpha_{mk} \vec{B}^{mk}(\vec{r})$$

$$B_r^{mk} = \frac{-1}{\sqrt{\mu^2 - k^2}} \left[k J_m'(\gamma) + \frac{m\mu}{\gamma} J_m(\gamma) \right] \sin(m\theta + kz)$$

$$B_\theta^{mk} = \frac{-1}{\sqrt{\mu^2 - k^2}} \left[\mu J_m'(\gamma) + \frac{mk}{\gamma} J_m(\gamma) \right] \cos(m\theta + kz)$$

$$B_z^{mk} = J_m(\gamma) \cos(m\theta + kz)$$

with $\gamma = r\sqrt{\mu^2 - k^2}$

Need to satify $B_r(r=a) = 0$

$$\int \vec{A} \cdot \vec{B} \, dV = K$$

$$\int B_z \, r dr d\theta = \psi$$

note: $B_r^{00} \equiv 0 \;\; \forall \mu$ and $\int B_z^{00} \, r dr d\theta \neq 0$

for $m \neq 0$ or $k \neq 0$

$$\left\{ \begin{array}{l} \int B_z^{mk} \, r dr d\theta = 0 \quad \text{and} \\ B_r^{mk}(r=a) = 0 \end{array} \right. \iff \mu = \mu^{mk}$$

since $\omega = \mu K$ for $\nabla \times \vec{B} = \mu \vec{B}$

$(\mu^{mk} a)_{min} = 3.11$ for $m = 1$ $ka = 1.25$

- for $0 < \theta < \frac{(\mu^{mk} a)_{min}}{2} = 1.6$

$\vec{B} = a_{00} \vec{B}^{00}$ with μa determined by

$$\frac{K}{\psi^2} = \frac{1}{2\pi a} \left[\frac{\mu a J_0^2(\mu a) + J_1^2(\mu a) - 2 J_0(\mu a) J_1(\mu a)}{J_1^2(\mu a)} \right]$$

$$\boxed{\begin{array}{l} \psi \text{ determines } a_{00} \\ \frac{K}{\psi^2} \text{ determines } \theta \text{ or } \mu a \end{array}}$$

- for $\theta = \frac{(\mu^{mk} a)_{min}}{2} = 1.6$

$\vec{B} = a_{00} \vec{B}^{00} + a_{mk} \vec{B}^{mk}$

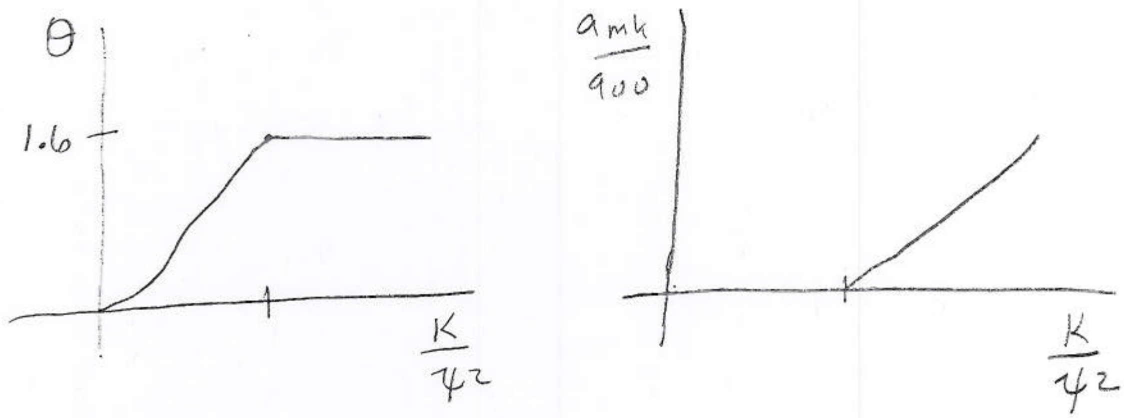
$$\boxed{\begin{array}{l} \psi \text{ determines } a_{00} \\ \frac{K}{\psi^2} \text{ determines } \frac{a_{mk}}{a_{00}} \end{array}}$$

Two major consequences

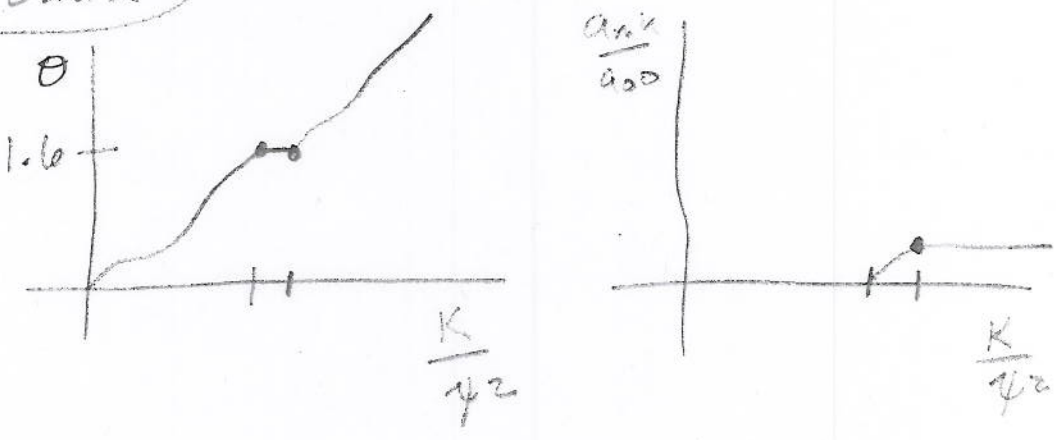
$\boxed{\#1}$ Current Limitation

axisymmetric state $0 < \theta < 1.6$

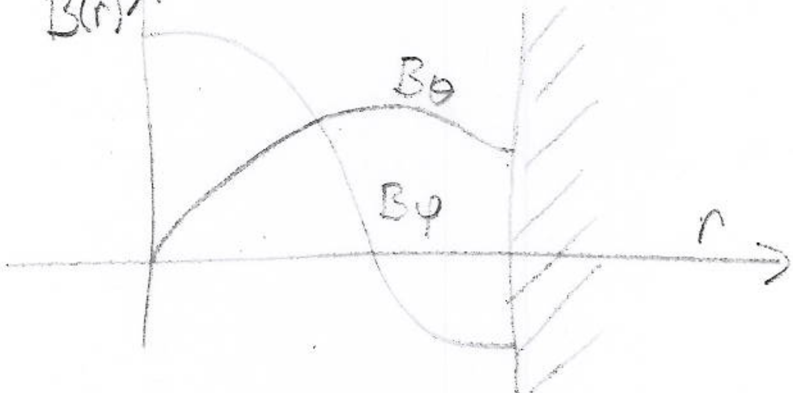
two states for $\theta = 1.6$



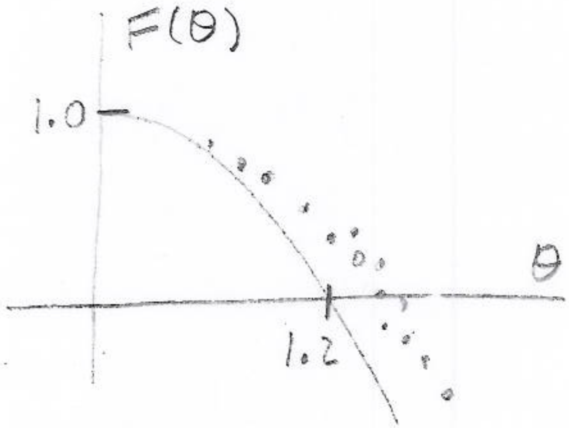
actual



#2 Relaxed state profile for no current limitation



$$B_\theta = \alpha J_1(\mu r)$$
$$B_\varphi = \alpha J_0(\mu r)$$
$$B_r = 0$$



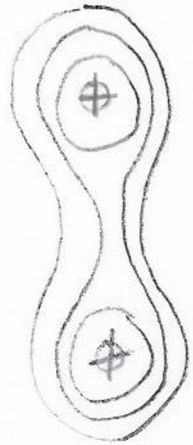
deviation due to
(1) finite wall η
(2) finite β

$$F(\theta) = \frac{J_0(\mu a)}{\langle B_\varphi/\alpha \rangle_r} = \frac{2\theta^2 J_0(2\theta)}{\int_0^{2\theta} J_0(x)\, x\, dx}$$

$$J_0(2.4) \approx 0$$

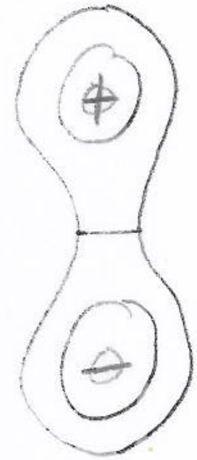
$$\Rightarrow \theta(F=0) = 1.2$$

Muti pinch



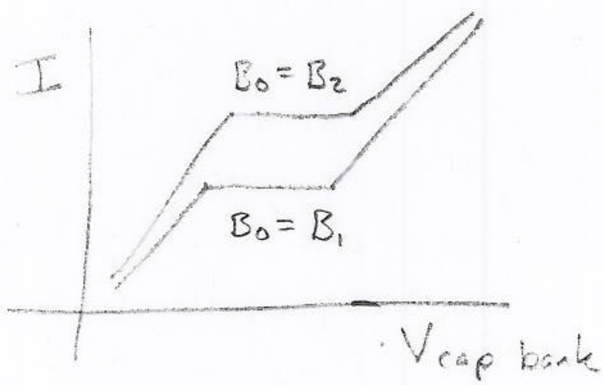
Relaxed profile

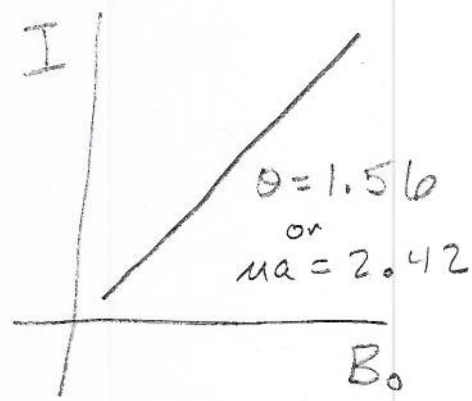
$\mu a = 1.5$



lowest eigen function

$\mu a = 2.21$





$$V_{cap\,bank} \sim \frac{K}{\psi^2}$$

$$\psi = \pi a^2 B_0$$

$$\theta \sim \frac{I}{B_0}$$

Helicity with Differential Geometry

Definitions:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\mathcal{J}^3 = i_{\left(\rho \frac{\partial}{\partial t} + \vec{J}\right)} vol^4$$

$$\mathfrak{F}^2 = \mathcal{E}^1 \wedge dt + \mathcal{B}^2 \qquad \mathcal{B}^2 = i_{\vec{B}}\, vol^3$$

$$\vec{u} = \gamma \frac{\partial}{\partial t} + \gamma \vec{v}$$

Maxwells Equations:

$$d\mathfrak{F}^2 = 0$$

$$d * \mathfrak{F}^2 = 4\pi \mathcal{J}^3$$

Scalar invariants:

$$\mathfrak{F}^2 \wedge \mathfrak{F}^2 = -2(\vec{E} \cdot \vec{B})\, vol^4$$

$$\mathfrak{F}^2 \wedge * \mathfrak{F}^2 = (E^2 - B^2)\, vol^4$$

Potential

since $d\mathfrak{F}^2 = 0 \qquad \mathfrak{F}^2 = da^1$

$$a^1 = \vec{A} \cdot d\vec{x} + \varphi\, dt$$

Helicity:

$$K^3 = a^1 \wedge da^1 = a^1 \wedge \mathfrak{F}^2$$
$$= i_{\left(\vec{A}\cdot\vec{B}\,\frac{\partial}{\partial t} + \vec{E}\times\vec{A} - \varphi\vec{B}\right)} vol^4$$

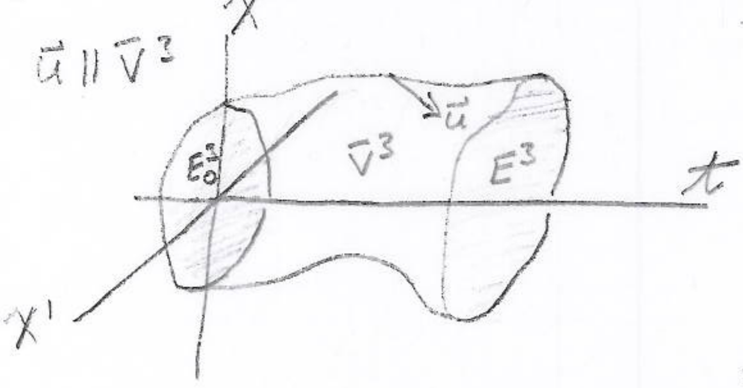
$\vec{E} + \vec{u}\times\vec{B} = 0$:

$$\mathcal{E}^1 = i_{\vec{u}}\, \mathfrak{B}^2$$
$$\mathfrak{F}\wedge\mathfrak{F} = 0$$
$$i_{\vec{u}}\, \mathfrak{F} = 0$$

Now lets prove helicity conservation

$$dK^3 = d(a^1\wedge da^1) = da^1 \wedge da^1$$
$$= \mathfrak{F}\wedge\mathfrak{F} = 0$$

Concider $\displaystyle 0 = \int_{V^4} dK^3 = \int_{\partial V^4} K^3 = \int_{\bar{V}^3} K^3 + \int_{E_0^3} K^3 + \int_{E^3} K^3$



$$= K - K_0 + \int_{\bar{V}^3} K^3$$

$$= K - K_0$$

$$\Rightarrow \quad K = K_0$$

$$\int_{\nabla^3} K^3 = \int_{\mathcal{U}\subset\mathbb{R}^3} K^3(\vec{u}, \vec{S}_1, \vec{S}_2) = \int_{\mathcal{U}} i_{\vec{u}}\, K^3(\vec{S}_1, \vec{S}_2)$$

$$\vec{u} = \gamma \frac{\partial}{\partial t} + \gamma \vec{v}$$

$$\vec{S}_1 = s_{1x} \frac{\partial}{\partial x} + S_{1y} \frac{\partial}{\partial y} + S_{1z} \frac{\partial}{\partial z}$$

$$\vec{S}_2 = s_{2x} \frac{\partial}{\partial x} + S_{2y} \frac{\partial}{\partial y} + S_{2z} \frac{\partial}{\partial z}$$

$$i_{\vec{u}} K^3 = i_u (A^1 \wedge dA^1) = (i_u A^1)\, dA^1 - A^1 \wedge (i_{\vec{u}} \mathcal{F}^2)$$

$$= (i_u A^1)\, \mathcal{F}^2$$

$$= \int_{\mathcal{U}} (i_u A^1)(\mathcal{E}^1 \wedge dt + \mathcal{B}^2)(\vec{S}_1, \vec{S}_2)$$

$$= 0$$

Plasma Parameters

$T_e \approx 100\,\text{eV}$

$n \approx 0.5 \times 10^{14}/\text{cm}^3$

$B \sim 1\,\text{kG}$

$I \sim 200\,\text{kA}$

$\tau_{\text{shell}} = 100\,\text{msec}$

$\tau_{\text{plasma}} = 10\,\text{msec} = \left(\frac{L}{c}\right)^2 \frac{1}{\eta}$

$\eta = 5 \times 10^{-17}\,\text{sec}$

$= \frac{\nu m}{n e^2} = \frac{\omega_p}{g}$

$\omega_{pe} = 4 \times 10^{11}/\text{sec} = 63\,\text{GHz}$

$\lambda_{de} = 10\,\mu\text{m}$

$\omega_{ce} = 2 \times 10^{10}/\text{sec} = 3\,\text{GHz}$

$r_{Le} = 300\,\mu\text{m}$

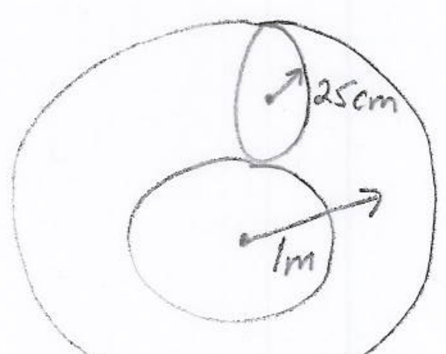
$g = 2 \times 10^{-5}$

$a = 0.2\,\mu\text{m}$

$b = 0.1\,\text{Å}$

$$\boxed{\begin{array}{l} r_{Le} \gg \lambda_d \gg a \gg b \\ \omega_{pe} \gg \omega_{ce} \end{array}}$$

$\begin{pmatrix} 10.102 \\ \text{Ichimaru} \end{pmatrix}$ $\nu_D^{(0)} = \frac{4(2\pi)^{1/2} n e^4}{3 m_e^2 \left(\frac{T_e}{m_e} + \frac{T_i}{m_i}\right)} \left(1 + \frac{T_i}{T_e}\right) \ln\Lambda \approx n \bar{v}_e b^2 \ln\Lambda \approx \nu_{ee}$



Some Stuff on MHD

$$\frac{d}{dt}\left[\int \vec{B}\cdot dS\right] = 0$$

flux tubes move with fluid

for equilibrium

$$\frac{d\vec{v}}{dt} = 0 \Rightarrow$$

$$\Rightarrow \quad \frac{\vec{J}\times\vec{B}}{c} = \nabla p$$

$$\Rightarrow \left\{ \begin{array}{l} \vec{B}\cdot\nabla p = 0 \\ J\cdot\nabla p = 0 \end{array} \right\}$$

$\vec{B}$ and $\vec{J}$ ∥ to surface of constant pressure

$$\nabla p = \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B}$$

$$= -\frac{1}{8\pi}\nabla_{\perp}(B^2) + \left(\frac{B^2}{8\pi}\right)(\hat{B}\cdot\nabla\hat{B})$$

$\nabla p =$

MHD Conditions

Quasi-Neutrality:

$L >> \lambda_D = 10 \mu m$

Ohm's Law:

- neglect $\vec{J} \times \vec{B}$ $\quad L^2 \omega >> \frac{c^2 \omega_{ce}}{\omega_{pe}^2} = 10^4 \frac{m^2}{sec}$
- neglect $\frac{\partial \vec{J}}{\partial t}$ $\quad L >> \frac{c}{\omega_p} = 1 mm$
- neglect ∇p $\quad L^2 \omega >> \frac{T_e/m}{\omega_{ce}} = 4 \times 10^3 \frac{m^2}{sec}$

Maxwell Eqs

- neglect $\frac{\partial \vec{E}}{\partial t}$ $\quad L\omega << c = 3 \times 10^8 m/sec$

Inverse Helicity Cascade

(Frisch, Pouquet, Léorat, Mazure 1975)

Full MHD:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\vec{v} = -(\vec{v}\cdot\nabla)\vec{v} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \nabla p + \vec{f}$$

$$\left(\frac{\partial}{\partial t} - \lambda\nabla^2\right)\vec{B} = -(\vec{v}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{v}$$

$$\nabla\cdot\vec{v} = \nabla\cdot\vec{B} = 0$$

Invariants: (if $\lambda = \nu = 0$ $\vec{f} = 0$)

$$E_T = \int(v^2 + B^2)$$

$$H_M = \int \vec{A}\cdot\vec{B}$$

$$H_c = \int \vec{v}\cdot\vec{B}$$

Can show: $H_M(k) \leq E_T(k)/k$

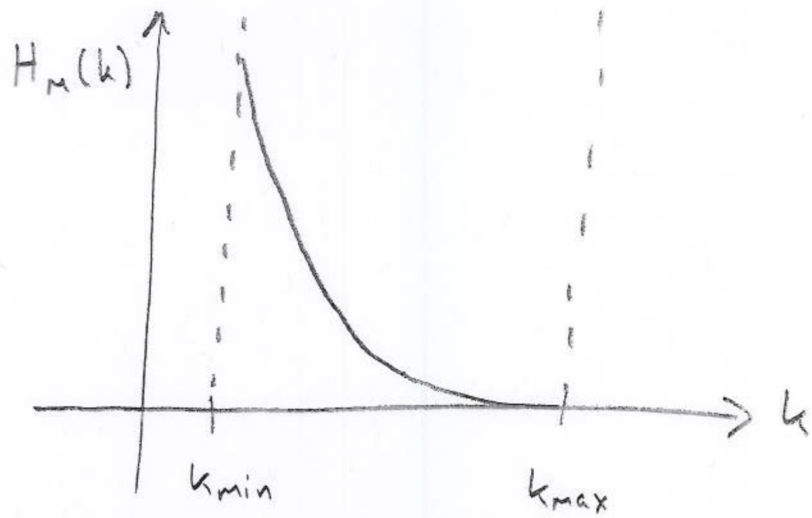
(realizability condition from pos-def of spectral tensor)

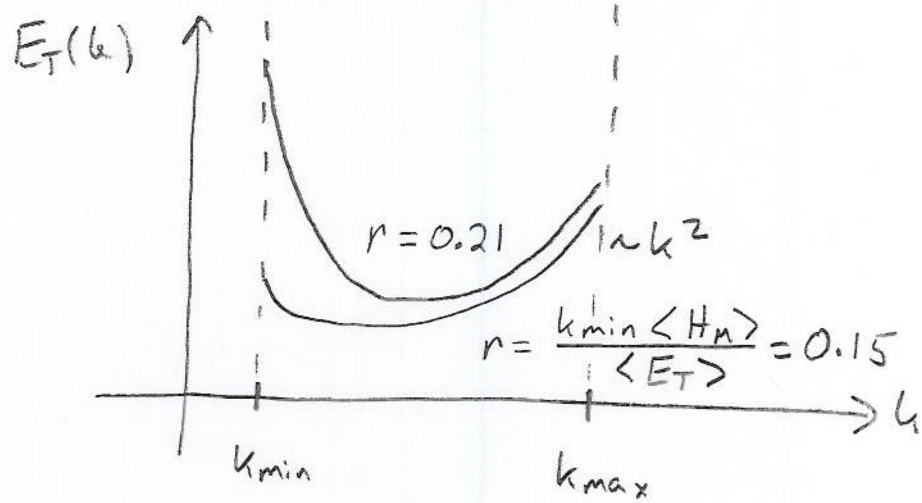
where $\langle E_T\rangle = \int E_T(k)\,dk$ ect.

Assume: $\rho = Z^{-1} e^{-\alpha E_T - \beta H_M - \gamma H_c}$

given $\alpha, \beta, \gamma, k_{min}, k_{max}$

can find $E_T(k)$, $H_M(k)$, ect





Conclusion:

If helicity and energy are injected at k_{min}, helicity will stay at k_{min} but energy will cascade to large k where it will be quickly dissipated.