Motion of Electron in Radial E-field Skew to Constant B-field

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GUIDING CENTER HAMILTONIAN

Use guiding center approximation leading to drift hamiltonian

$$H = \frac{p_z^2}{2m} - e\,\phi\left(x\,, \frac{c\,p_x}{eB}\,, z \right)$$

Assume cylindrically symmetric E-field skewed in x-z plane to a constant B-field in the z-direction by an angle β (see figure 1). That is

$$\phi(x,y,z) = \phi_o\, f\left[\frac{\sqrt{\left(x \cos\beta - z \sin\beta \right)^2 + y^2}}{x_o} \right]$$

Apply the following extended cononical transformation

$$\overline{H} = \frac{H}{m\,\Omega^2\, x_o^2} \qquad \tau = \Omega\, t \qquad \frac{d\overline{p}_i}{d\tau} = -\frac{\partial \overline{H}}{\partial \overline{x}_i} \qquad \frac{d\overline{x}_i}{d\tau} = \frac{\partial \overline{H}}{\partial \overline{p}_i} \qquad \text{(eq. of motion)}$$

$$\overline{x} = \frac{x \cos\beta - z \sin\beta}{x_o} \qquad \overline{p}_x = \frac{p_x \cos\beta - p_z \sin\beta}{m\,\Omega\, x_o}$$

$$\overline{z} = \frac{z \cos\beta + x \sin\beta}{x_o} \qquad \overline{p}_z = \frac{p_z \cos\beta + p_x \sin\beta}{m\,\Omega\, x_o}$$

This gives the hamiltonian

$$\overline{H}\left(\overline{x}\,, \overline{p}_x\,, \overline{p}_z \right) = \frac{1}{2}\left(\overline{p}_z \cos\beta - \overline{p}_x \sin\beta \right)^2 - \overline{b}^2 f\left[\overline{x}^2 + \left(\overline{p}_x \cos\beta + \overline{p}_z \sin\beta \right)^2 \right]$$

where $\overline{b}^2 = \dfrac{e\,\phi_o}{m\,\Omega^2\, x_o^2} = \left(\dfrac{\omega_p}{\Omega} \right)^2 A^2$ $\qquad A^2 = \dfrac{1}{4\pi} \dfrac{\phi_o / \left(e/\lambda_o \right)}{\left(x_o/\lambda_o \right)^2} \sim O(1)$

$$\omega_p^2 = \frac{4\,\pi\, e^2}{m\, \lambda_o^3}$$

There are two constants of the motion, $\overline{H}$, since the hamiltonian is time translation invariant, and, $\overline{p}_z$, since the hamiltonian is $\overline{z}$ translation invariant. Some values of ϕ and A for various density profiles is shown in table 1. X-Y orbits about a wire, uniform density, and n=exp(-R) are plotted on the attached graphs.

EXISTENCE OF CLOSED ORBITS

A restriction on the range of $\bar{b}$ where closed orbits can exist can be obtained by examining the kronecker index of the vector field about singular points of the vector field. This can be done by the following reasoning. Let a vector field be defined on $\mathbb{R}^2$. Suppose there exists a closed bounded orbit, defining a closed manifold M^1. This orbit encloses k singular points of the vector field, P_k. The index of this vector field about this curve will be +1 (see figure 2). Since the sum of the indexes about the singular points enclosed by the curve will equal the index of the vector field about the curve, there must be at least on P_i with index greater than zero. Conversely, if there does not exist a P_i with index greater than zero there can not be any bounded orbits.

Now let's examine the index of the vector field to see what this implies for the dynamics being analyzed. Solve the Hamiltonian for $\bar{x}^2$

$$\bar{x}^2 = -\left(\bar{p}_x \cos\beta + \bar{p}_z \sin\beta \right)^2 + g\left[\frac{1}{\bar{b}^2}\left(-\bar{H} + \frac{1}{2}\left(\bar{p}_z \cos\beta - \bar{p}_x \sin\beta \right)^2 \right)\right]$$

where

$$g = \left(f^{-1} \right)^2$$

Now expand g to first order

$$\bar{x}^2 \approx -\left(\bar{p}_x \cos\beta + \bar{p}_z \sin\beta \right)^2 + g\left(\xi_o \right) + \frac{d\,g\left(\xi_o \right)}{d\,\xi}\left(\xi - \xi_o \right)$$

where

$$\xi = \frac{1}{\bar{b}^2}\left(-\bar{H} + \frac{1}{2}\left(\bar{p}_z \cos\beta - \bar{p}_x \sin\beta \right)^2 \right)$$

This equation can be written in the form

$$\bar{x}^2 + a\left(\bar{p}_x + \frac{b}{2\,a} \right)^2 \approx \left(\frac{b^2}{4\,a} - c \right)$$

with

$$a = \cos^2\beta - \frac{\sin^2\beta}{2\,B^2}$$

$$b = \bar{p}_z \sin\beta \cos\beta \left(2 + \frac{1}{B^2} \right)$$

$$c = \bar{p}_z^2 \left(\sin^2\beta - \frac{\cos^2\beta}{2\,B^2} \right) + \frac{\bar{H}}{B^2} + \xi_o \frac{d\,g\left(\xi_o\right)}{d\,\xi} - g\left(\xi_o\right)$$

$$B^2 = \frac{\bar{b}^2}{\dfrac{d\,g\left(\xi_o\right)}{d\,\xi}}$$

This is the equation of hyperbola (index=-1) or an ellipse (index=+1) depending on the value of a. Obviously if $a<0$ for all points no closed orbits can exist. Therefore if closed orbits exist

$$\tan\beta < B\sqrt{2} = \bar{b}\left(\frac{f'\left(\eta\right)}{\eta} \right)_{max}^{\frac{1}{2}} = \frac{\omega_p}{\Omega} \frac{1}{\sqrt{4\,\pi}} \left(\frac{\lambda_o^2 E_\rho / e}{\rho / \lambda_o} \right)_{max}^{\frac{1}{2}} = \frac{<\omega_p>_{max}}{\Omega\sqrt{2}}$$

$$\text{where} \quad \left(\frac{<\omega_\rho>}{\omega_p} \right)^2 = \frac{2}{\rho^2} \int_0^\rho \lambda_o^3\, n\left(r\right)\, r\, dr$$

A SIMPLE WAY TO EXAMINE CLOSURE

Imagine an electron executing an $\mathbf{E}_\perp \times \mathbf{B}$ orbit about a wire skew to a constant B-field (see figure 3). The time for the electron to execute one of these orbits is

$$\tau_d \sim \frac{1}{\omega_d} = \frac{\rho}{v_d} = \frac{\rho\,B}{c\,E_\rho \cos\beta}$$

At the same time there is a parallel acceleration to B

$$a_{\parallel \text{ to } B} = \frac{e}{m}\left(E_{\rho} \sin\beta \right)$$

The component of this acceleration perpendicular to the wire is

$$a_{\parallel B \perp \text{wire}} = \left[\frac{e}{m}\left(E_{\rho} \sin\beta \right) \right] \sin\beta$$

The criteria we shall impose to insure closure is that the distance that the electron travels due to this acceleration perpendicular to the wire should be less than the distance the particle is from the wire.

$$a_{\parallel B \perp \text{wire}} \; \tau_d^2 \lesssim \rho$$

This gives the following condition on $\tan\beta$

$$\tan\beta \lesssim \frac{\omega_\rho}{\Omega} \sqrt{\frac{E_\rho / (e/\lambda_o^2)}{\rho / \lambda_o}}$$

This approximate condition is the same as the condition obtained through the analysis of the dynamics.

Figure 1.

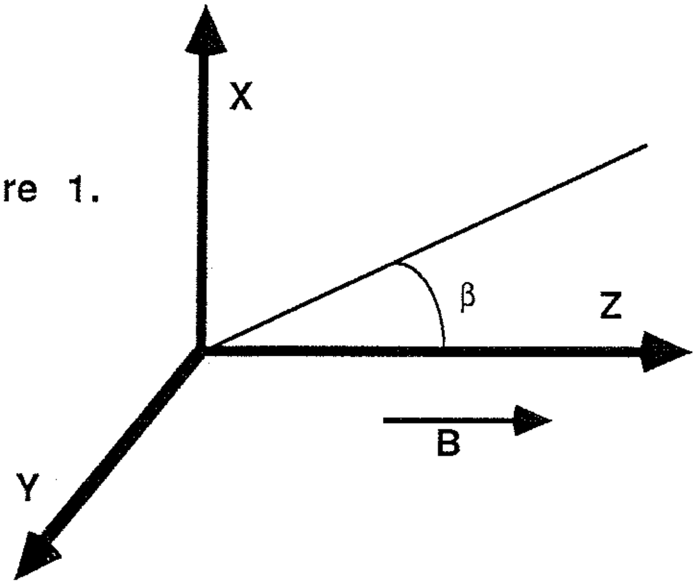
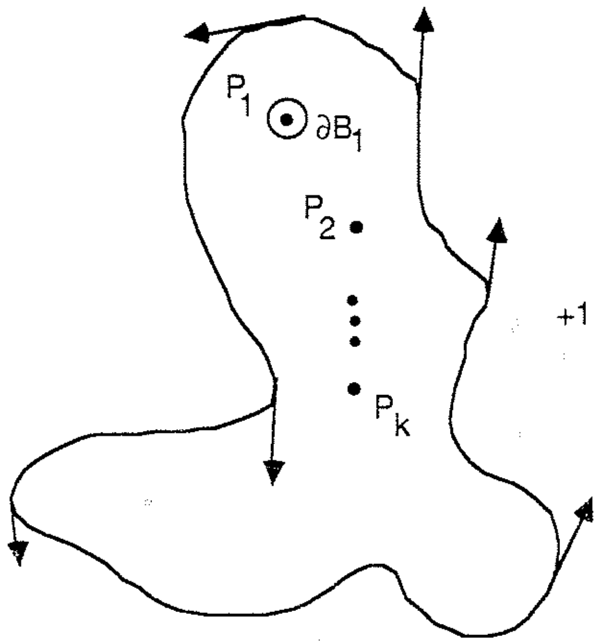


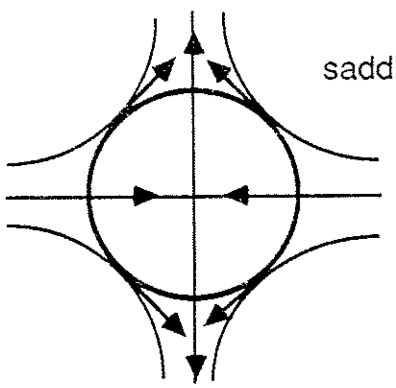
Table 1.

$n(r)$	$\phi(r)$	$f(\eta)$	A	x_o
$\frac{\delta(r)}{\lambda_o r}$	$\frac{e}{\lambda_o}\ln\left(r/\lambda_o\right)^2$	$\ln\left(\eta^2\right)$	$\frac{1}{\sqrt{4\pi}}$	λ_o
$\frac{1}{\lambda_o^2 r}$	$\frac{4\pi e}{\lambda_o}\left(\frac{r}{\lambda_o}\right)$	η	1	λ_o
$\frac{1}{\lambda_o^3}$	$\frac{\pi e}{\lambda_o}\left(\frac{r}{\lambda_o}\right)^2$	η^2	$\frac{1}{2}$	λ_o
$\frac{r}{\lambda_o^4}$	$\frac{4\pi}{9}\frac{e}{\lambda_o}\left(\frac{r}{\lambda_o}\right)^3$	η^3	$\frac{1}{3}$	λ_o
$\frac{e^{-\frac{r}{r_0}}}{\lambda_o^3}$	$\frac{4\pi e r_o^2}{\lambda_o^3}\left[e^{-\frac{r}{r_o}} - 1 + \mathrm{Ein}\left(\frac{r}{r_o}\right)\right]$ where $\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t}dt$	$e^{-\eta} - 1 + \mathrm{Ein}(\eta)$ $\approx \ln(\eta) \quad (\eta > 1)$ $\approx \left(\frac{\eta}{2}\right)^2 \quad (\eta < 1)$	1	r_o

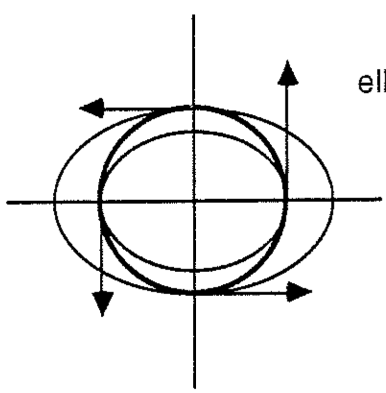
Figure 2.



$$+1 = \text{index of } \mathbf{v} \text{ on } M = \sum_{i=1}^{k} \text{index on } \partial B_i$$

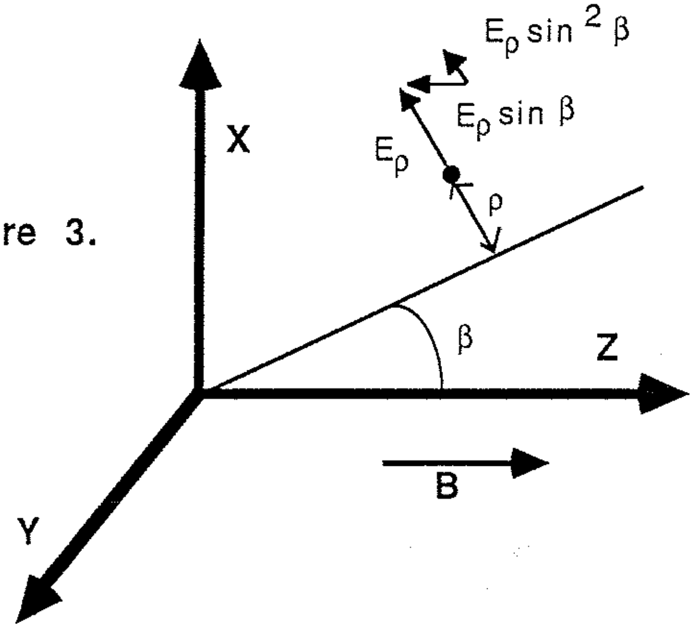


saddle ==> index = -1

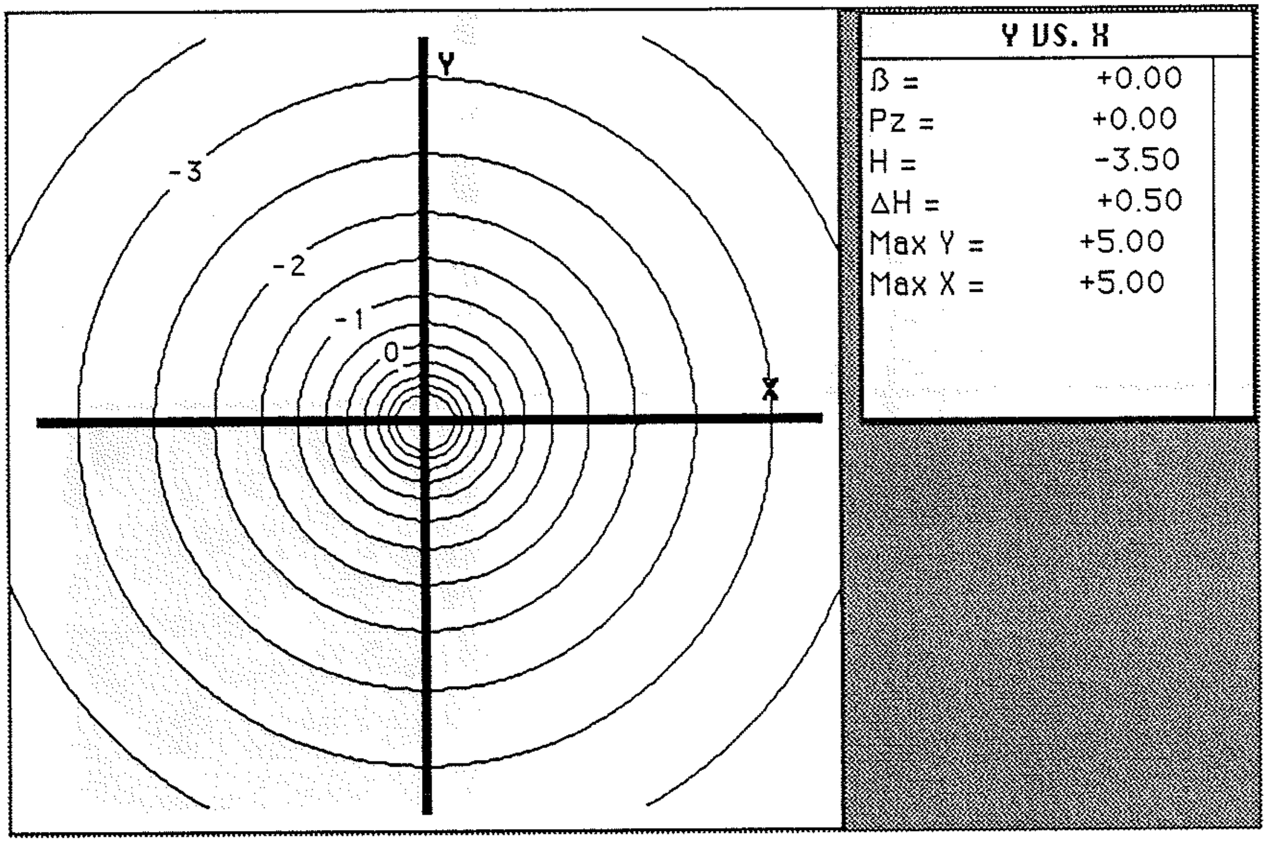


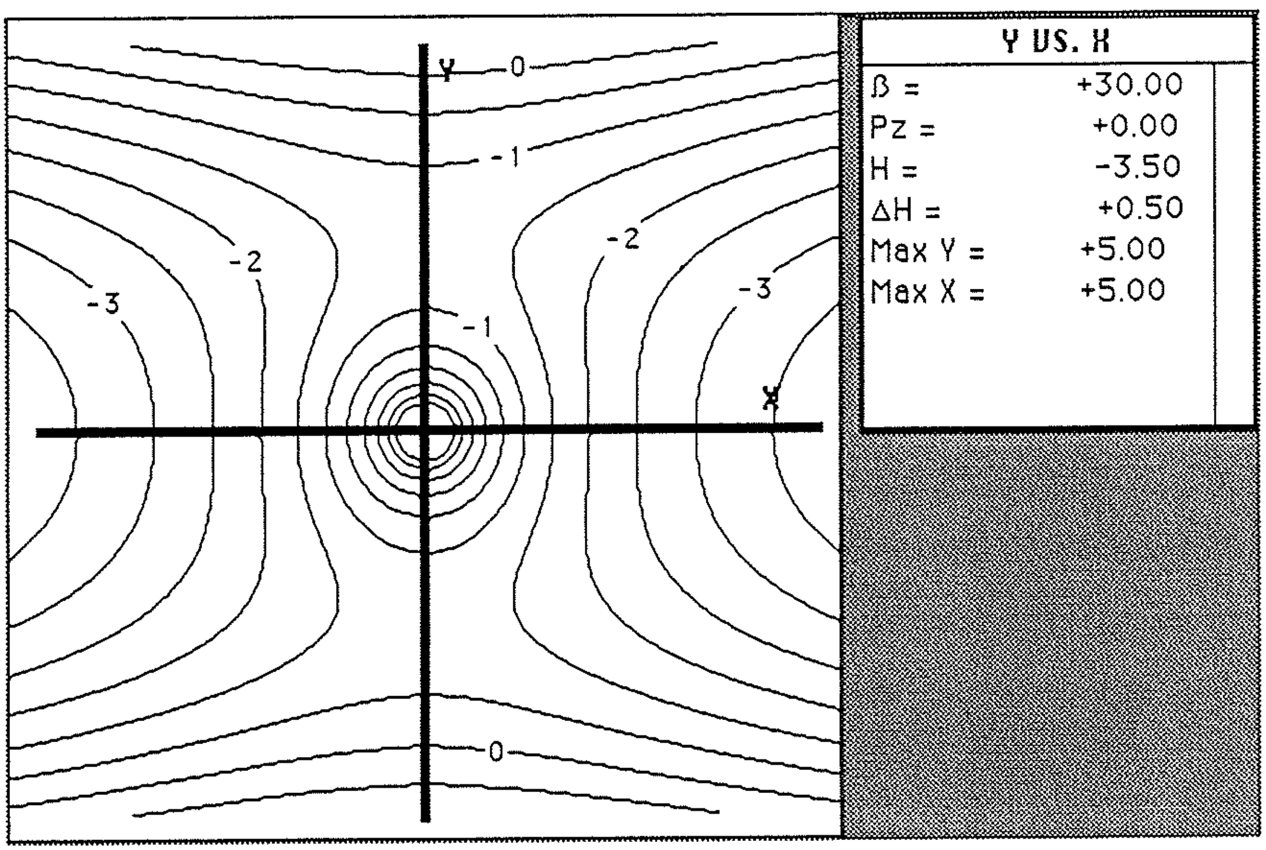
ellipse ==> index = +1

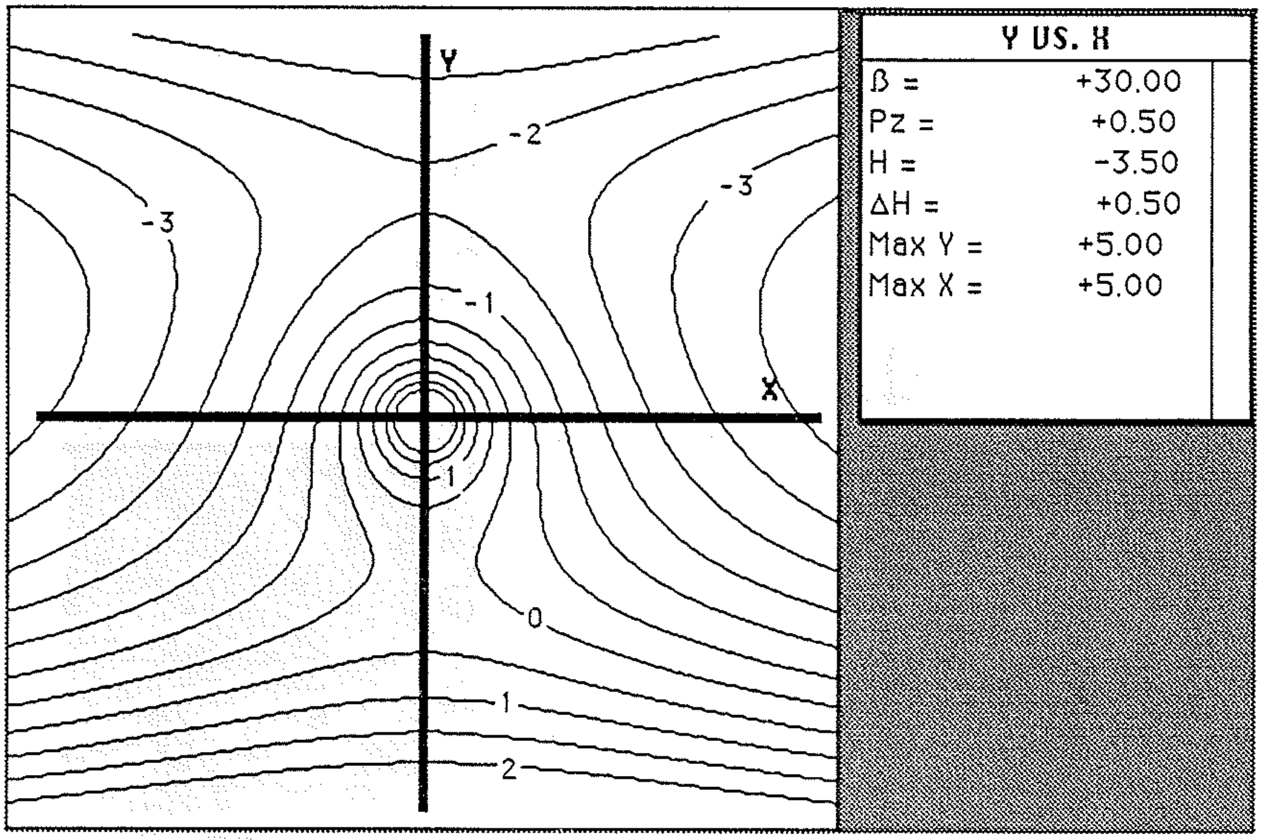
Figure 3.



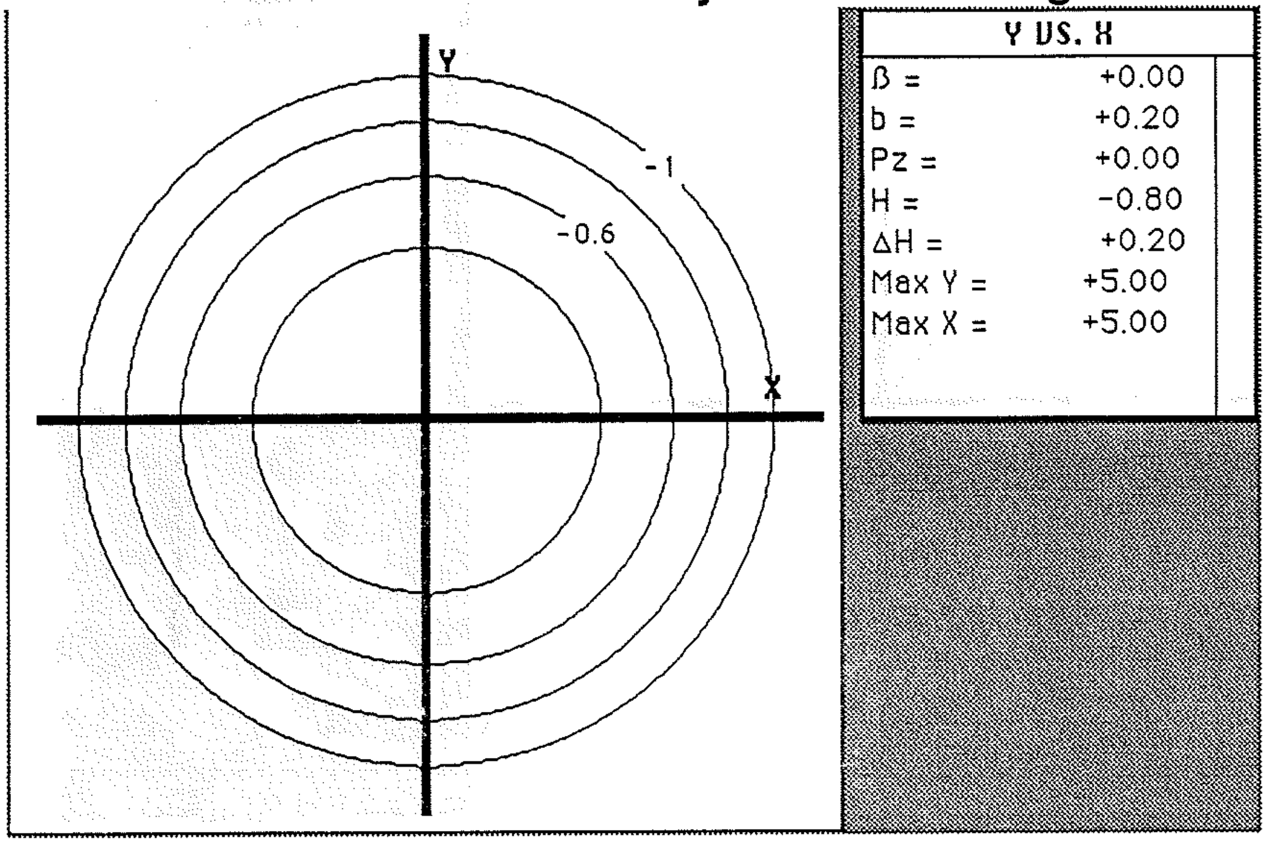
Case 1. Wire

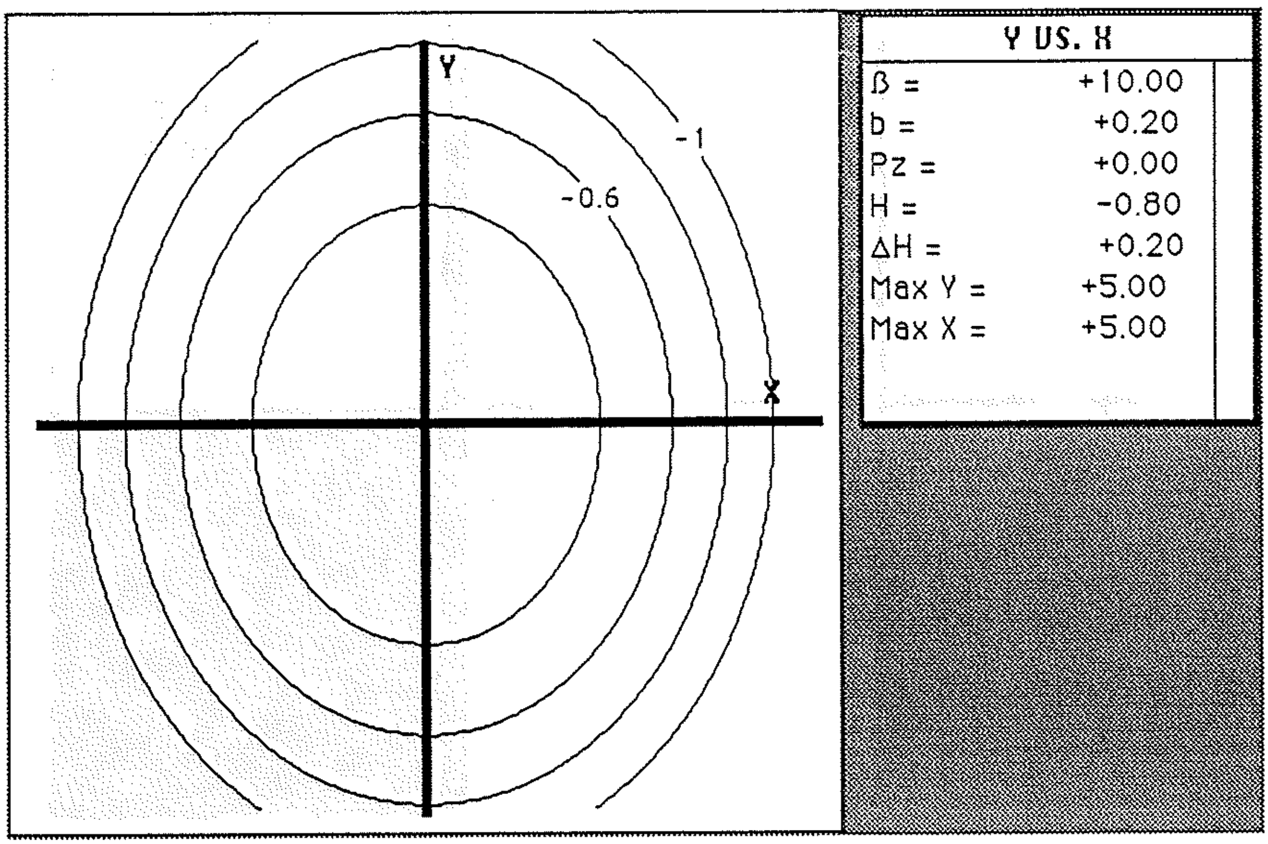

Y
X
-3
-2
-1
0
Y VS. X
ß = +0.00
Pz = +0.00
H = -3.50
ΔH = +0.50
Max Y = +5.00
Max X = +5.00

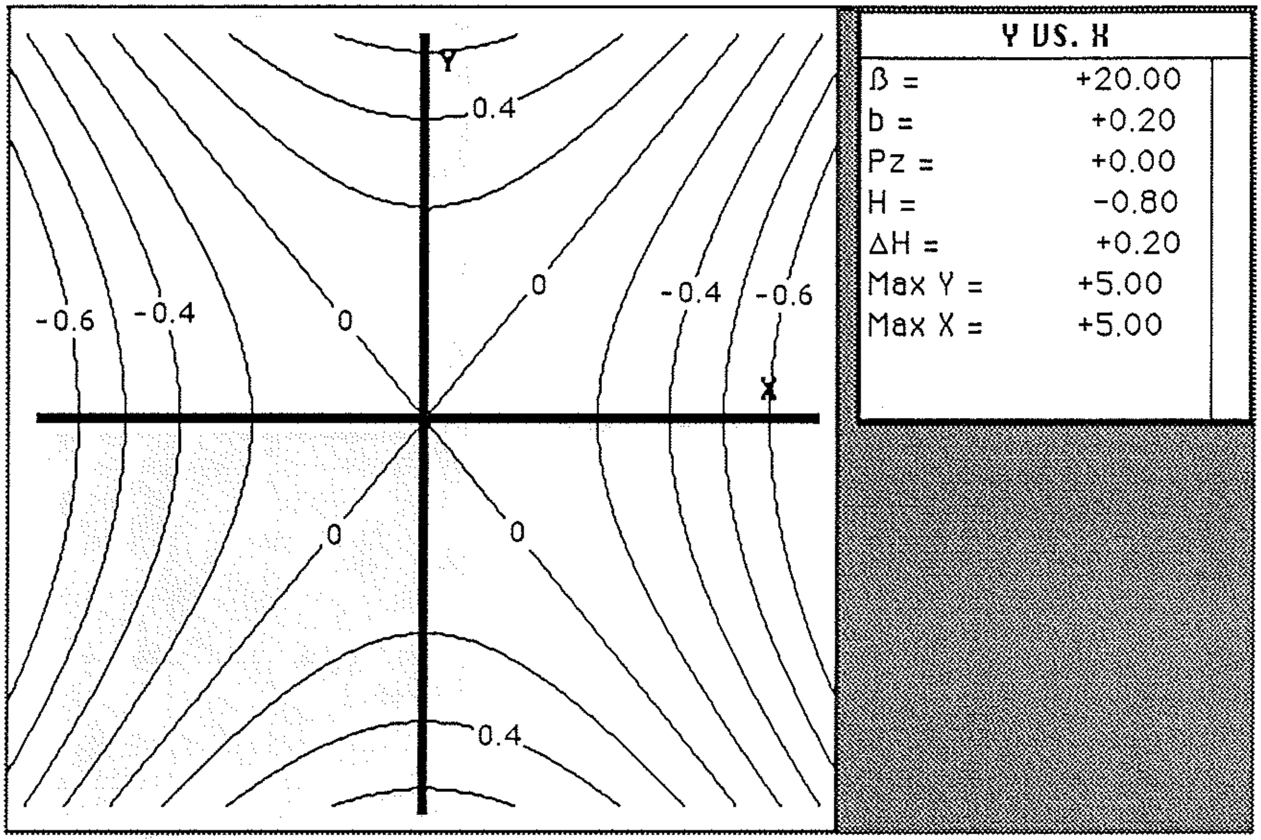

Y
X
0
-1
-2
-3
-2
-3
-1
0
Y VS. X
ß = +30.00
Pz = +0.00
H = -3.50
ΔH = +0.50
Max Y = +5.00
Max X = +5.00

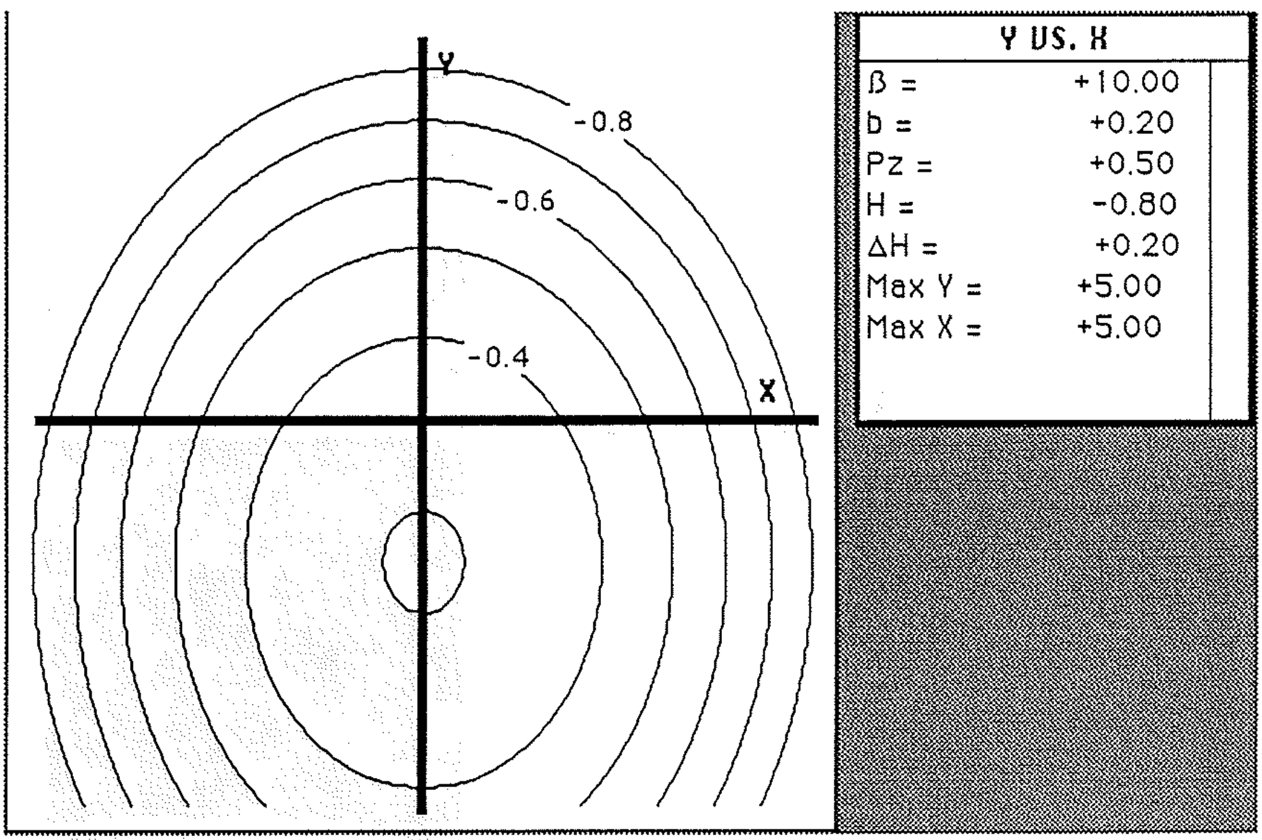

Y VS. X
ß = +30.00
Pz = +0.50
H = -3.50
ΔH = +0.50
Max Y = +5.00
Max X = +5.00
Y
X
-3
-2
-1
0
1
2

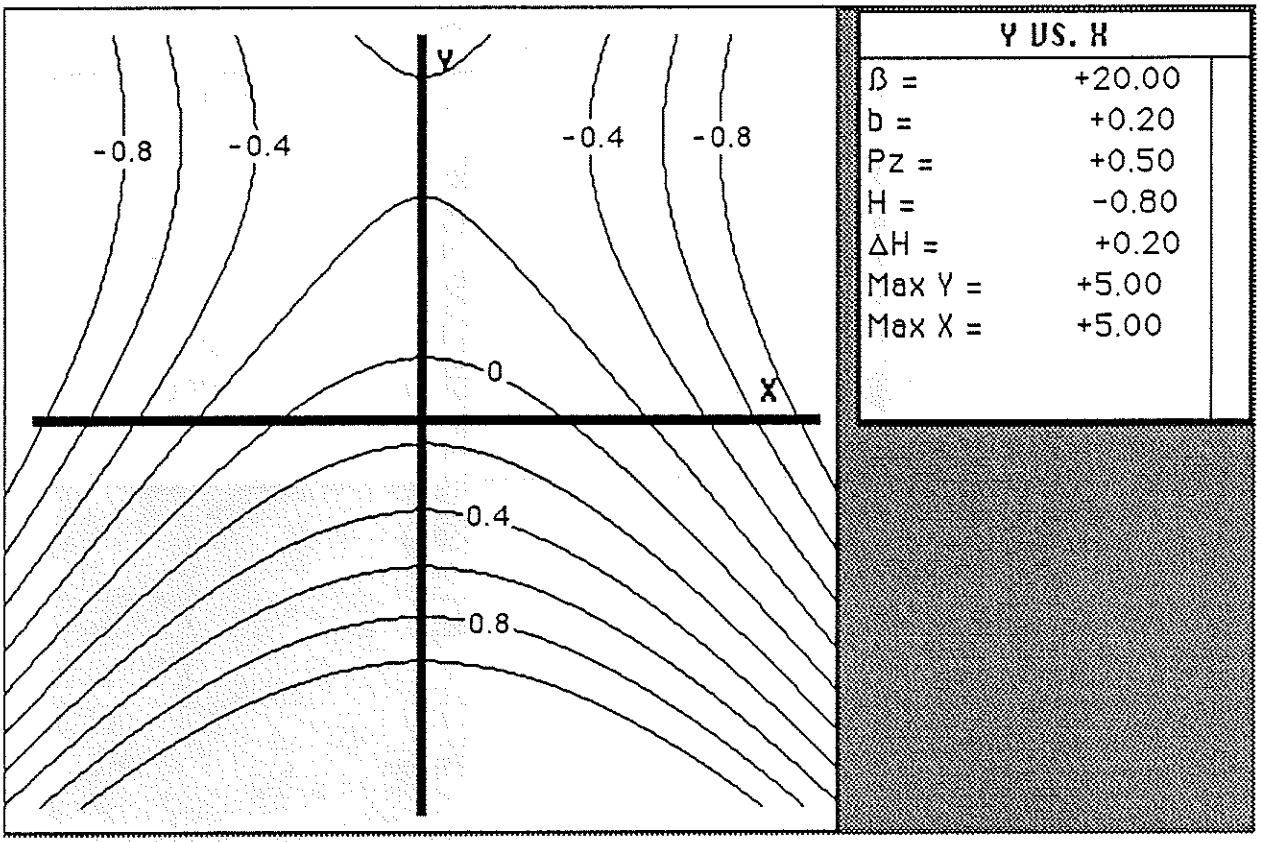
Case 2. Uniform Cylindrical Charge



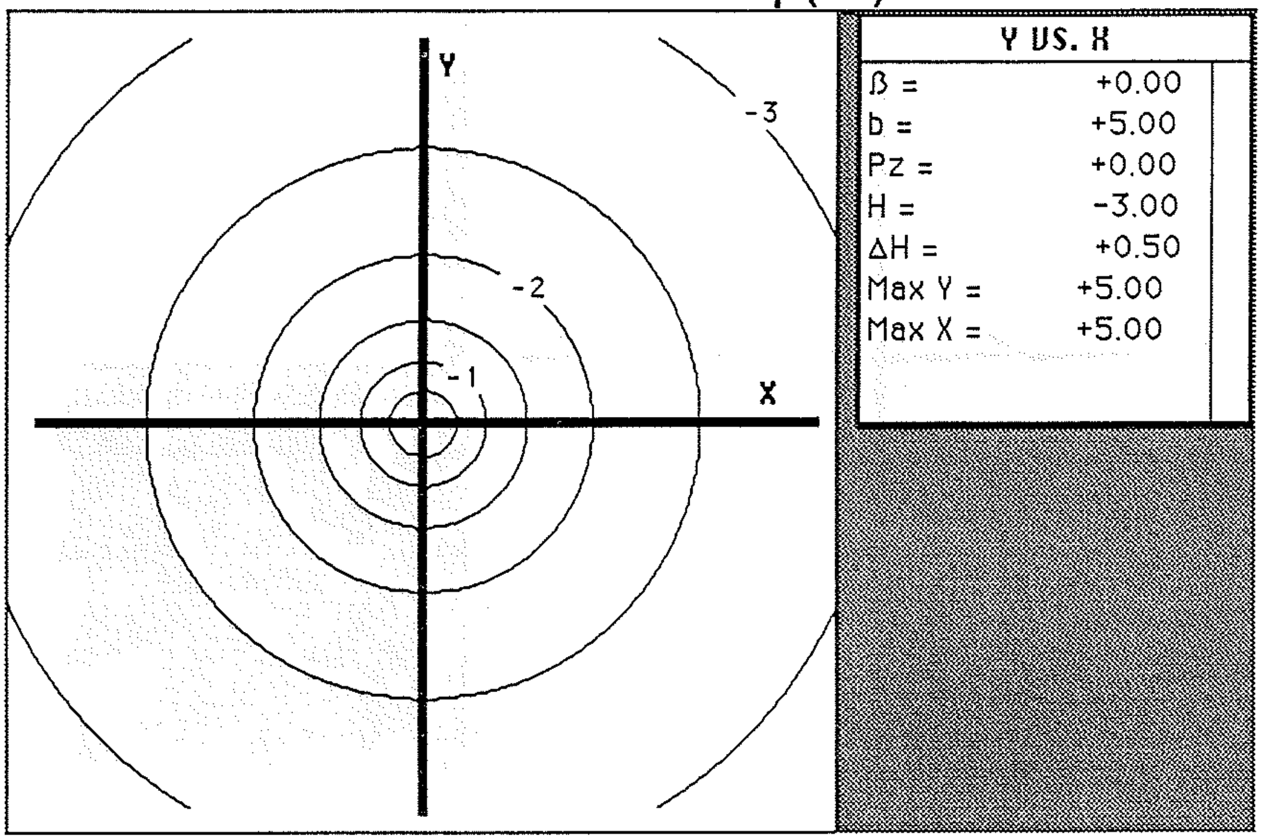


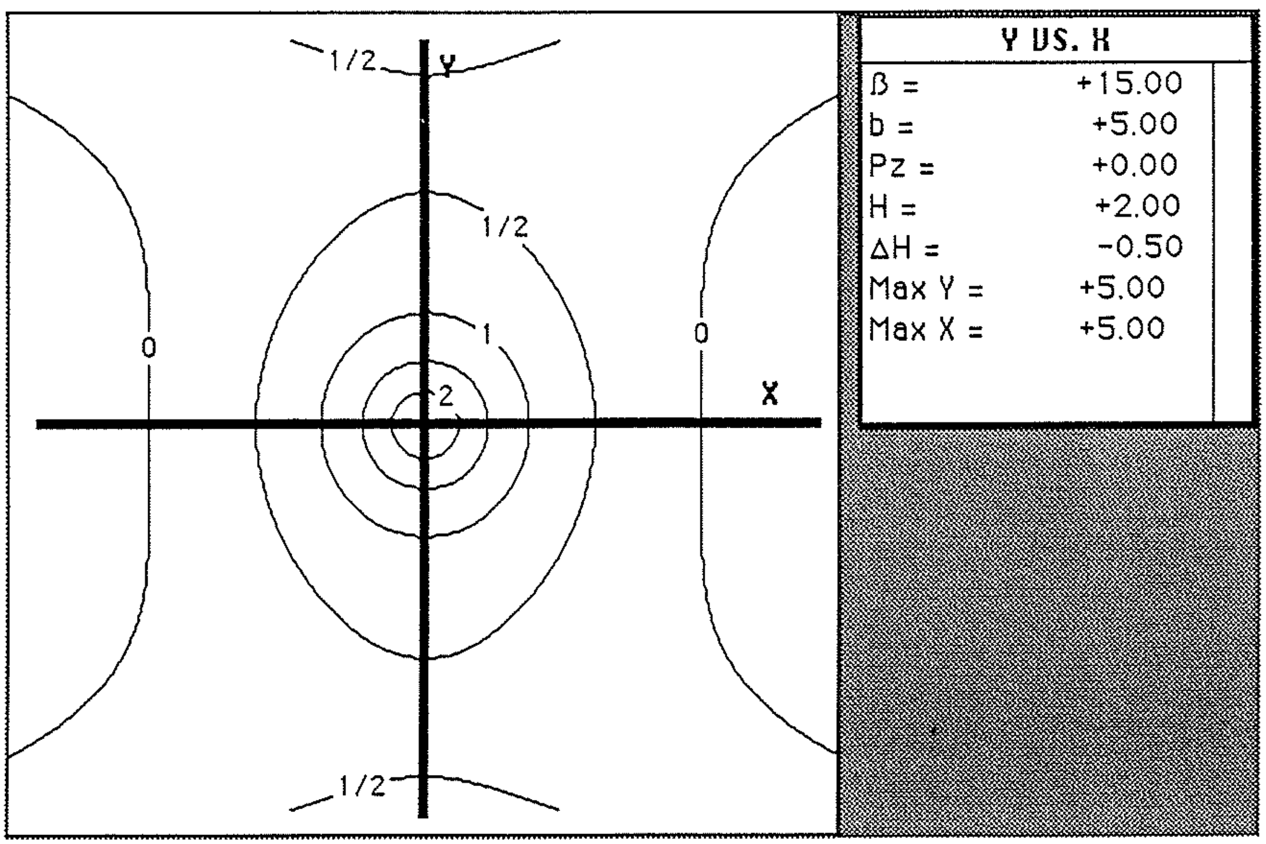

Y
X
0.4
0
-0.4
-0.6
Y VS. X
ß = +20.00
b = +0.20
Pz = +0.00
H = -0.80
ΔH = +0.20
Max Y = +5.00
Max X = +5.00

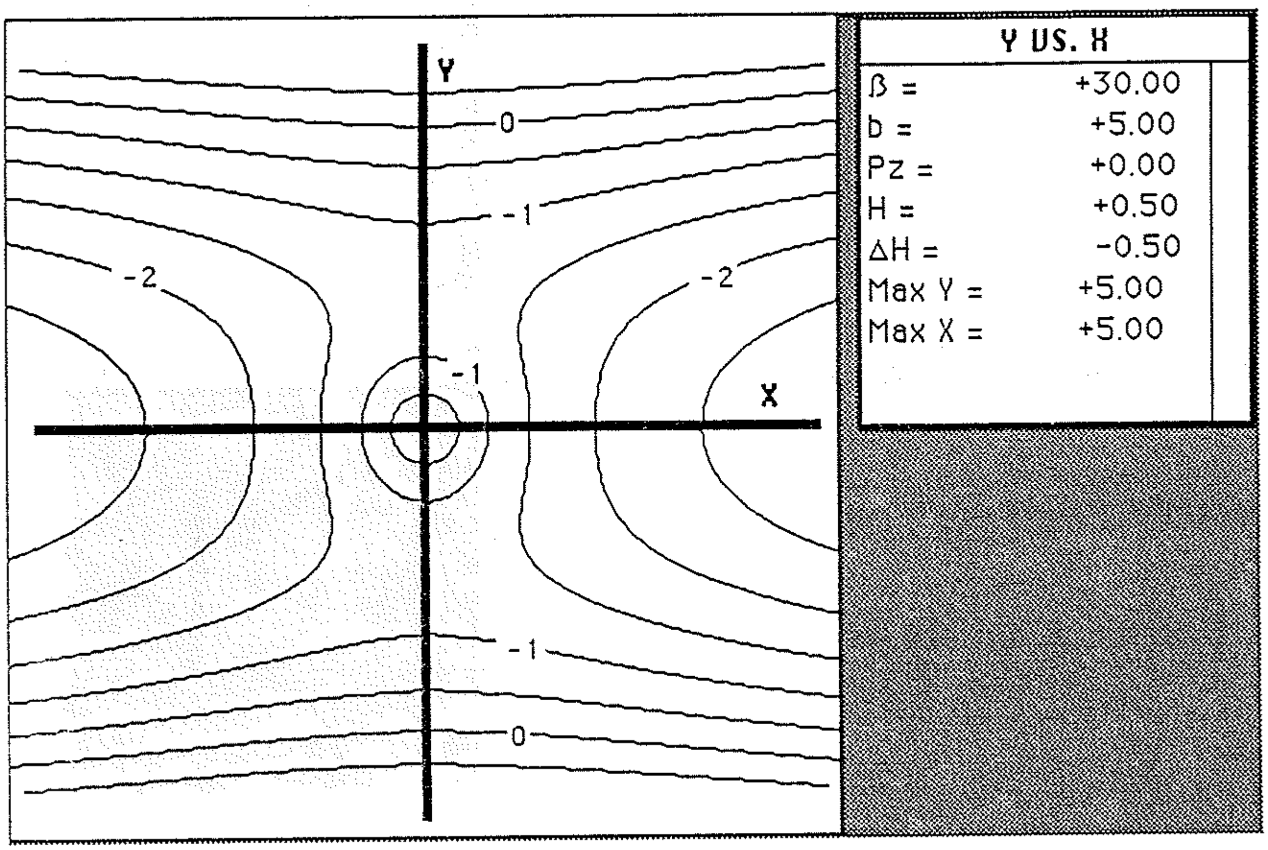

Y
X
-0.8
-0.6
-0.4
Y VS. X
ß = +10.00
b = +0.20
Pz = +0.50
H = -0.80
ΔH = +0.20
Max Y = +5.00
Max X = +5.00

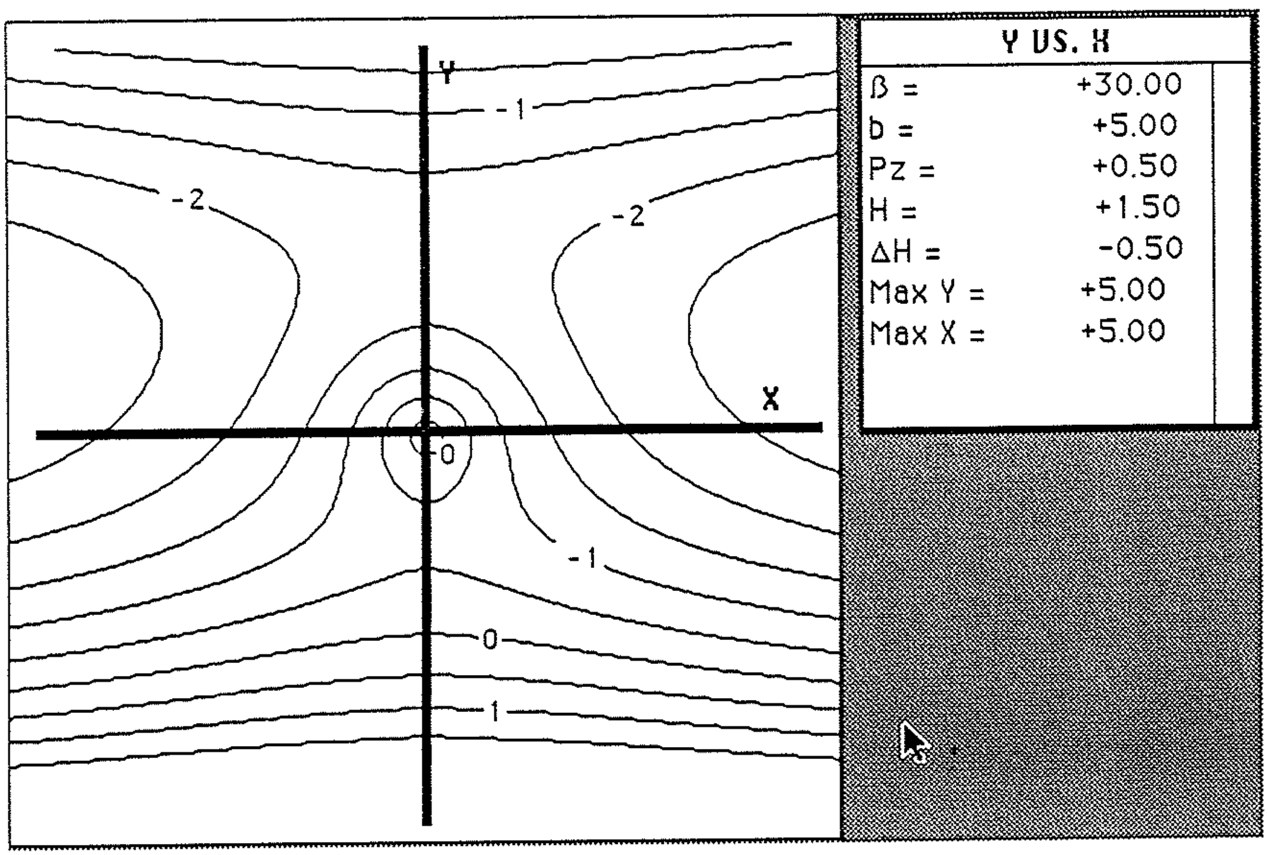

-0.8
-0.4
-0.4
-0.8
Y
X
0
0.4
0.8
Y VS. X
ß = +20.00
b = +0.20
Pz = +0.50
H = -0.80
ΔH = +0.20
Max Y = +5.00
Max X = +5.00

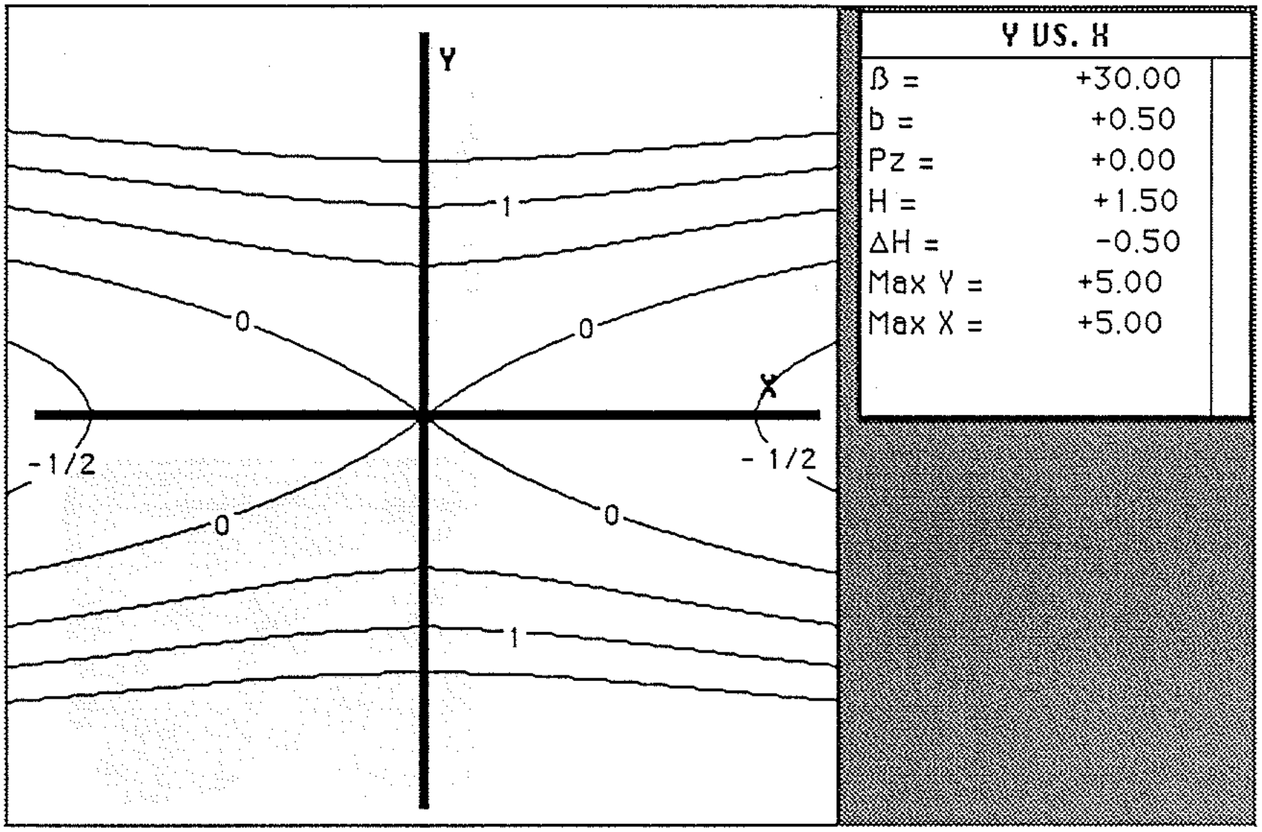
Case 3. n=exp(-R)


Y
X
-3
-2
-1
Y VS. X
ß = +0.00
b = +5.00
Pz = +0.00
H = -3.00
ΔH = +0.50
Max Y = +5.00
Max X = +5.00

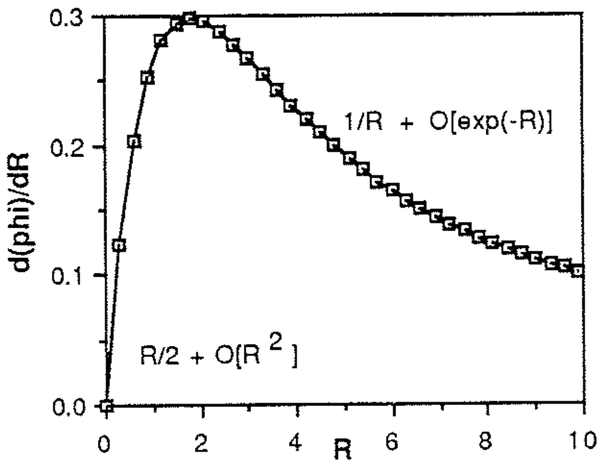

1/2
Y
X
1/2
1
2
0
0
1/2
Y VS. X
ß = +15.00
b = +5.00
Pz = +0.00
H = +2.00
ΔH = -0.50
Max Y = +5.00
Max X = +5.00


Y
X
0
-1
-2
-2
-1
-1
0
Y VS. X
ß = +30.00
b = +5.00
Pz = +0.00
H = +0.50
ΔH = -0.50
Max Y = +5.00
Max X = +5.00


Y
X
-1
-2
-2
0
-1
0
1
Y VS. X
ß = +30.00
b = +5.00
Pz = +0.50
H = +1.50
ΔH = -0.50
Max Y = +5.00
Max X = +5.00


Y
X
1
0
0
-1/2
- 1/2
0
0
1
Y VS. X
ß = +30.00
b = +0.50
Pz = +0.00
H = +1.50
ΔH = -0.50
Max Y = +5.00
Max X = +5.00

E-field for n = exp(-R)



Potential for n=exp(-R)