Collisional Equipartition Rate for a Magnetized Pure Electron Plasma

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presented at the
1990 American Physical Society Meeting of the
Division of Plasma Physics

[M. E. Glinsky, T. M. O'Neil, M. N. Rosenbluth, K. Tsuruta, and S. Ichimaru, Bul. Am. Phys. Soc. **35**, 2134 (1990)]

Abstract*

The collisional equipartition rate between the parallel and perpendicular velocity components of a weakly correlated electron plasma that is immersed in a uniform magnetic field is calculated. Here, parallel and perpendicular refer to the direction of the magnetic field. The rate depends on the ratio r_c/b, where $r_c=\sqrt{T/m}/\Omega_c$ is the cyclotron radius and $b=e^2/T$ is the classical distance of closest approach. For a strongly magnetized plasma (i.e., $r_c/b<<1$), the equipartition rate is exponentially small ($\sim\exp[-2.35(r_c/b)^{-2/5}]$).[1] For a weakly magnetized plasma (i.e., $r_c/b>>1$), the rate is the same as for an unmagnetized plasma except that r_c/b replaces λ_D/b in the Coulomb logarithm.[2] (It is assumed here that $r_c<\lambda_D$; for $r_c>\lambda_D$, the plasma is effectively unmagnetized.) This paper presents a numerical treatment that spans the intermediate regime $r_c/b\sim1$, connects on to asymptotic results in the two limits $r_c/b<<1$ and $r_c/b>>1$, and is in good agreement with recent experiments (see poster by B. Beck et al.). Also an improved asymptotic expression for the rate in the high field limit is given.

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*Supported by NSF grant PHY87-06358, DOE grant DEFG03-88ER53275, the San Diego Supercomputer Center, and an NSF Graduate Fellowship.

[1]T.M. O'Neil and P.G. Hjorth, Phys. Fluids **28**, 3241 (1985).

[2]D. Montgomery, G. Joyce and L. Turner, Phys. Fluids **17**, 2201 (1974).

Definition of the Equipartition Rate

Consider a pure electron plasma that is immersed in a uniform magnetic field $\vec{B} = B\,\hat{z}$. Let the velocity distribution be of the form

$$f(\vec{v}) = \left(\frac{m}{2\pi T_{||}}\right)^{1/2} \left(\frac{m}{2\pi T_{\perp}}\right) \exp\left[-\frac{m v_{||}^2}{2 T_{||}} - \frac{m v_{\perp}^2}{2 T_{\perp}}\right]$$

where

$$| T_{||} - T_{\perp} | << T_{||}, T_{\perp} .$$

The collisional <u>equipartition rate</u> ν is defined through the equation

$$\frac{d T_{\perp}}{d t} = \nu \left(T_{||} - T_{\perp} \right) .$$

Previous Results

1. Strongly magnetized plasma

$$\left(r_c \ll b \ , \ r_c \equiv \frac{\bar{v}}{\Omega_c} \ , \ b \equiv \frac{e^2}{T} \ , \ \bar{v} \equiv \sqrt{\frac{T}{m}} \right)$$

$$\nu = n \bar{v} b^2 \ I\left(\frac{r_c}{b} \right)$$

$$I\left(\frac{r_c}{b} \right) \sim \exp\left[-2.35 \left(\frac{b}{r_c} \right)^{2/5} \right] \quad *$$

2. Weakly magnetized plasma

$$\left(b \ll r_c \ll \lambda_D \right)$$

$$I \sim \ln\left(\frac{r_c}{b} \right) \quad ‡$$

3. Effectively unmagnetized plasma

$$\left(\lambda_D < r_c \right)$$

$$I \sim \ln\left(\frac{\lambda_D}{b} \right)$$

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Binary Collision

$$\frac{d\vec{v}_1}{dt} + \Omega_c \vec{v}_1 \times \hat{z} = \frac{e^2}{m} \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$\frac{d\vec{v}_2}{dt} + \Omega_c \vec{v}_2 \times \hat{z} = \frac{e^2}{m} \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3}$$

Center of mass motion and relative motion decouple.

$$\left(\vec{V} \equiv \frac{\vec{v}_1 + \vec{v}_2}{2} \;,\; \vec{v} \equiv \vec{v}_2 - \vec{v}_1 \;,\; \vec{r} \equiv \vec{r}_2 - \vec{r}_1 \;,\; \mu \equiv \frac{m}{2} \right)$$

$$\frac{d\vec{V}}{dt} + \Omega_c \vec{V} \times \hat{z} = 0$$

$$\frac{d\vec{v}}{dt} + \Omega_c \vec{v} \times \hat{z} = \frac{e^2}{\mu} \frac{\vec{r}}{|\vec{r}|^3}$$

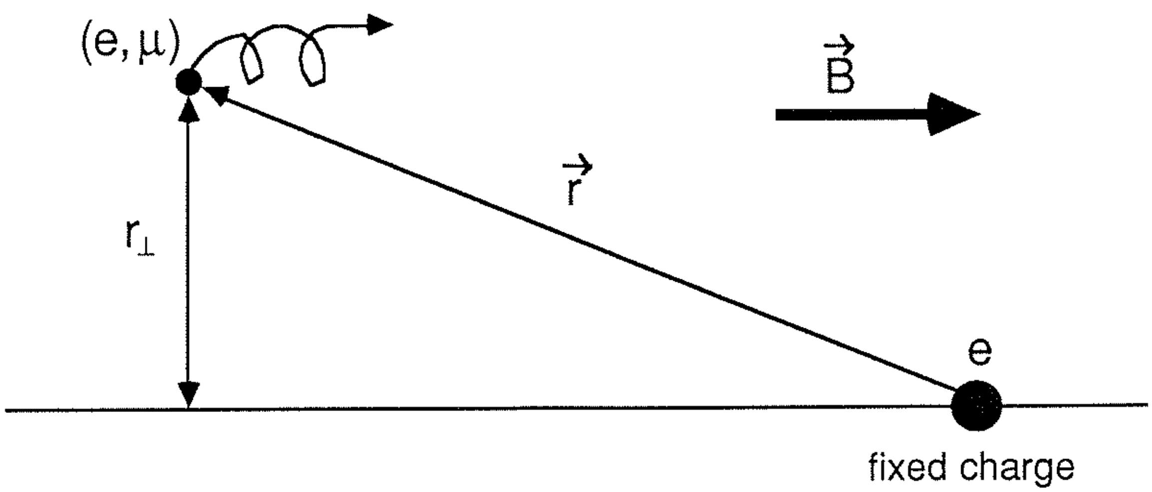
$$E_\perp = \frac{m\, v_{1\perp}^2}{2} + \frac{m\, v_{2\perp}^2}{2} = \frac{\mu\, v_\perp^2}{2} + \frac{2m\, V_\perp^2}{2}$$

$$E_\parallel = \frac{m\, v_{1\parallel}^2}{2} + \frac{m\, v_{2\parallel}^2}{2} = \frac{\mu\, v_\parallel^2}{2} + \frac{2m\, V_\parallel^2}{2}$$

Center of mass motion is unchanged during collision, therefore

$$\Delta E_\perp = -\Delta E_\parallel = \Delta\left(\frac{\mu\, v_\perp^2}{2}\right)$$

Relative Motion



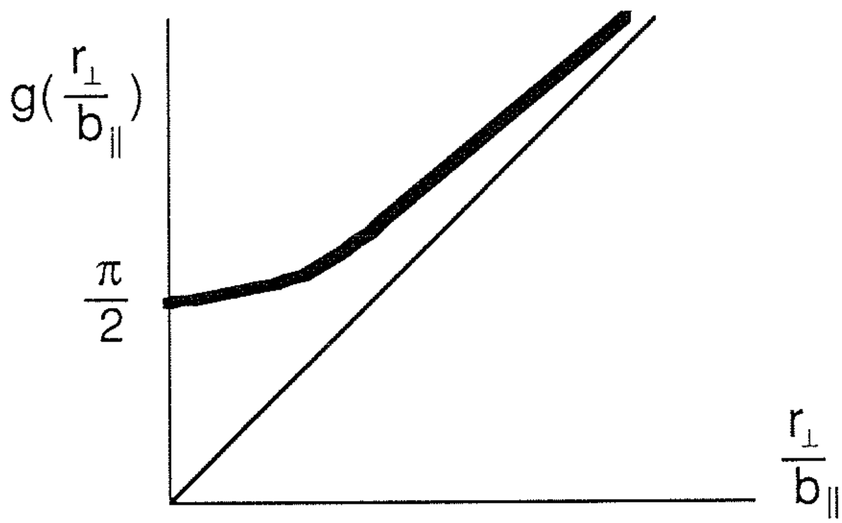
An <u>adiabatic invariant</u> exists in the limit

$$\Omega_c \tau_c >> 1,$$

where τ_c is the duration of a collision.

$$\Delta E_\perp \sim \exp\left(-\Omega_c \tau_c\right) \sim \exp\left\{-\Omega_c\left[\frac{b_\parallel}{v_\parallel} g\left(\frac{r_\perp}{b_\parallel}\right)\right]\right\}$$

where $b_\parallel \equiv e^2 \Big/ \left(\frac{\mu v_\parallel^2}{2}\right)$.



In a plasma, the adiabatic invariant produces a <u>dynamical shielding</u> (for the exchange of parallel and perpendicular kinetic energy).

$$\left|\vec{r}_1 - \vec{r}_2\right|_{min} > r_c = \frac{\bar{v}}{\Omega_c} \quad \Rightarrow \quad \Omega_c \tau_c > 1$$

$$\Rightarrow \text{ dynamical shielding}$$

Therefore, we can use a Boltzmann-like collision operator, provided that $r_c < \lambda_D$. The Boltzmann operator omits Debye shielding (recall Landau's derivation of the Fokker-Planck operator from the Boltzmann operator; Debye shielding was included in an *ad hoc* fashion), but nothing is lost if the dynamical screening length is shorter than the Debye length.

A <u>Boltzmann-like operator</u> leads to an <u>integral expression for the rate</u>

$$\frac{\partial f(\vec{v}_1, t)}{\partial t} = \int_0^\infty 2\pi r_\perp dr_\perp \int d^3\vec{v}_2 \; |v_{2\|} - v_{1\|}| \left[f(\vec{v}_2') f(\vec{v}_1') - f(\vec{v}_2) f(\vec{v}_1) \right]$$

$$\frac{\partial T_\perp}{\partial t} = \int d^3\vec{v}_1 \; \frac{m v_{1\perp}^2}{2} \frac{\partial f(\vec{v}_1, t)}{\partial t}$$

(use center of mass, relative velocities, and detailed balance)

$$\nu = \frac{n}{4T^2} \int_0^\infty 2\pi r_\perp dr_\perp \int d^3\vec{v} \; |v_\|| \left[\Delta\left(\frac{\mu v_\perp^2}{2} \right) \right]^2 f_r(\vec{v})$$

$$f_r(\vec{v}) = \left(\frac{\mu}{2\pi T} \right)^{3/2} \exp\left(-\frac{\mu v^2}{2T} \right)$$

$$T_\perp \approx T_\| = T$$

Monte Carlo Evaluation of the Equipartition Rate

Make a change of variables to the dimensionless

$$\vec{\eta} \equiv \frac{\vec{r}}{2b} \quad , \quad \vec{u} \equiv \frac{\vec{v}}{\sqrt{2}\,\bar{v}} \quad , \quad \tau \equiv \frac{\bar{v}}{\sqrt{2}\,b}\, t$$

and write the collision rate as

$$I\left(\frac{r_c}{b}\right) = \frac{\nu}{n\,\bar{v}\,b^2}$$

$$= 2\sqrt{2}\,\pi \int_0^{\infty} d\eta_\perp\, \eta_\perp \int_{-\infty}^{\infty} du_{||} \int_0^{2\pi} d\psi \int_0^{\infty} du_\perp\, u_\perp\, |\,u_{||}\,| \left[\Delta\left(\frac{u_\perp^2}{2}\right)\right]^2 \frac{e^{-u^2/2}}{(2\pi)^{3/2}}$$

where we have used cylindrical coordinates for $\vec{u}$.

In the expression for $I\left(\frac{r_c}{b}\right)$, $\Delta\left(\frac{u_\perp^2}{2}\right)$ is a function of $(u_\perp, u_{||}, \psi, \eta_\perp)$ determined by integration of the equations of motion

$$\frac{d\vec{u}}{d\tau} + \left(\frac{\sqrt{2}\,b}{r_c}\right)\vec{u}\times\hat{z} = \frac{1}{2}\,\frac{\vec{\eta}}{\eta^3} \quad , \qquad \frac{d\vec{\eta}}{d\tau} = \vec{u}$$

over the course of a collision.

To more efficiently do the integral for ν, we change coordinates from $\left(u_\perp, u_\parallel, \psi, \eta_\perp\right)$ to $\left(x_1, x_2, x_3, x_4\right)$ defined by

$$x_1 = \frac{1}{A_1}\int_0^{u_\parallel} du_\parallel \int_0^\infty d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_2 = \frac{1}{A_2}\int_0^{\eta_\perp} d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_3 = \frac{1}{A_3}\int_0^{u_\perp} du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$x_4 = \frac{1}{A_4}\int_0^{\psi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

where

$$A_1 = \int_0^\infty du_\parallel \int_0^\infty d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_2 = \int_0^\infty d\eta_\perp \int_0^\infty du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_3 = \int_0^\infty du_\perp \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

$$A_4 = \int_0^{2\pi} d\psi \;\; W(u_\parallel, u_\perp, \psi, \eta_\perp)$$

One can easily show that the Jacobian for this transformation is

$$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_\parallel, u_\perp, \psi, \eta_\perp)} = \frac{W(u_\parallel, u_\perp, \psi, \eta_\perp)}{A_1}$$

The equipartition rate can now be written as

$$I\left(\frac{r_c}{b}\right) = \frac{2A_1}{\sqrt{\pi}} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 dx_4 \; \frac{u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2}}{W(u_{\|}, u_{\perp}, \psi, \eta_{\perp})} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2$$

To make the Monte Carlo integration most efficient we would like to choose

$$W(u_{\|}, u_{\perp}, \psi, \eta_{\perp}) \sim u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2$$

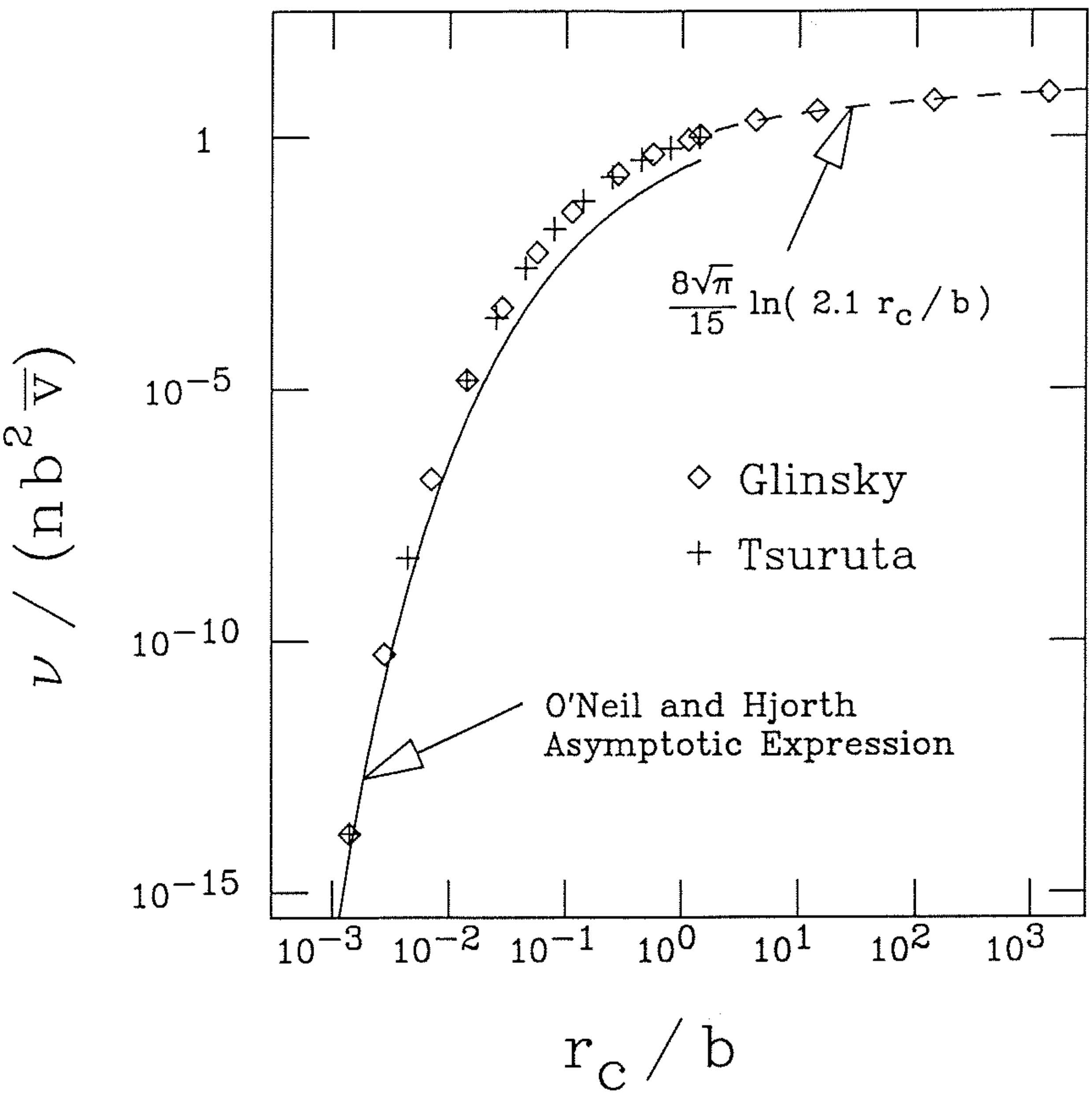
so that the integrand is reasonably uniform over the whole domain of integration.

We estimate the value of the integrand by picking N (x_1, x_2, x_3, x_4) points from a uniform distribution for each x_i between 0 and 1. We integrate the equations of motion using a Bulirsch-Stoer technique to find $\Delta\left(\frac{u_{\perp}^2}{2}\right)$. The equipartition rate is then

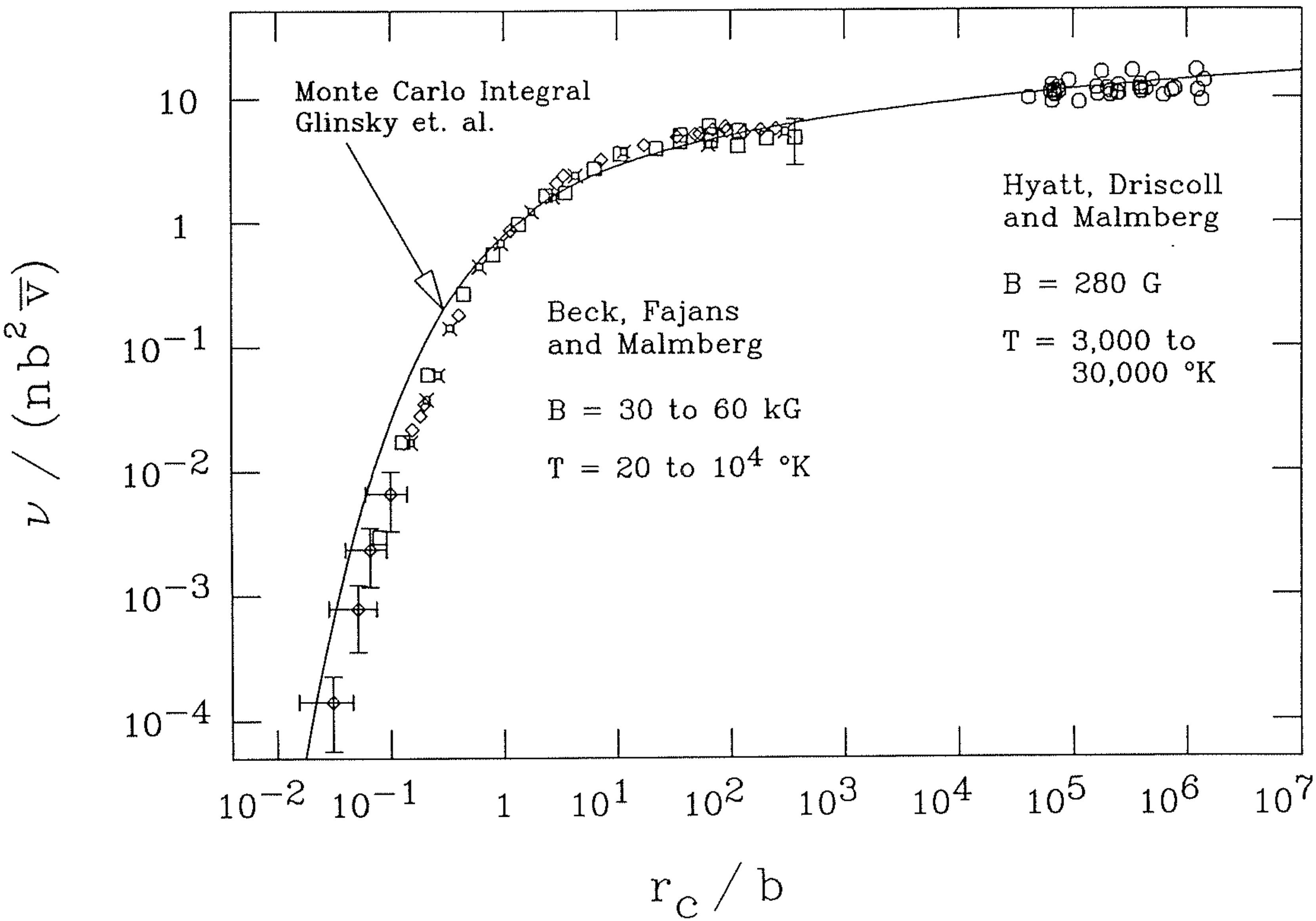
$$I\left(\frac{r_c}{b}\right) \approx \frac{2A_1}{\sqrt{\pi}} \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{u_{\|}\, u_{\perp}\, \eta_{\perp}\, e^{-u^2/2}}{W(u_{\|}, u_{\perp}, \psi, \eta_{\perp})} \left[\Delta\left(\frac{u_{\perp}^2}{2}\right)\right]^2 \right\}_i$$

A second Monte Carlo calculation was done using a rejection method to generate the initial configurations. The equations of motion were integrated using a 4th order Runge-Kutta scheme.

Graph of Monte Carlo Results



Monte Carlo Results Compared to Experiment



Analytic Expression for the Equipartition Rate

One can write the change in perpendicular energy during a collision as an asymptotic series for $\Delta\left(\frac{u_\perp^2}{2}\right)$ in the limit $u_\parallel^3 \frac{r_c}{b} << 1$. When the series is substituted into the integral for ν, the following expression was obtained by O'Neil and Hjorth

$$I(\bar{\varepsilon}) \approx (2.48)\, \bar{\varepsilon}^{1/5} \exp\left(-E\, \bar{\varepsilon}^{-2/5}\right) \quad \textbf{for} \quad \bar{\varepsilon} << 1$$

where

$$\bar{\varepsilon} \equiv \frac{r_c}{b} \quad \text{and} \quad E \equiv \frac{5}{6}(3\pi)^{2/5}\, 2^{1/5} \approx 2.35 \quad .$$

A more detailed evaluation by Rosenbluth gives

$$I(\bar{\varepsilon}) \approx \left[(10.2)\, \bar{\varepsilon}^{\,7/15} + (86.0)\, \bar{\varepsilon}^{\,11/15}\right] \exp\left(-E\, \bar{\varepsilon}^{-2/5}\right)$$

Monte Carlo Results Compared to Asymptotic Expressions for $r_c/b<<1$